Replicating Portfolios in Algo Risk

Using the Algo Risk Optimizer to replicate a set of scenario dependent liability cash flows

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Algorithmics

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INTRODUCTION

The main idea of a replicating portfolio is to find a portfolio of assets whose value is equal to the value of a liability portfolio under today's market conditions and future market conditions. Because simulating a replicating portfolio consisting of a few hundred assets is more computationally efficient than simulating the entire liability portfolio, the replicating portfolio is used in place of the liability portfolio when performing any number of risk analyses. Replicating portfolios are often used to calculate risk measures such as value-at-risk or cash-flow-at-risk, as well as for economic capital and regulatory capital calculations. In addition, replicating portfolios can be used in the construction of dynamic hedge programs or as performance targets for asset managers.

This paper provides a high-level overview of the replicating portfolio construction process available within Algo Risk. Some key differentiating characteristics of Algo Risk include the broad asset coverage, a flexible optimization framework, and an interactive risk dashboard for constructing the optimization problem and for analyzing the resulting replicating portfolio(s).

APPLICATIONS OF REPLICATING PORTFOLIOS

Creating a replicating portfolio process can be split into three main tasks. Each of these tasks is addressed in more detail later in this paper:

1. **pre-optimization** – creation of the economic scenarios and corresponding scenario dependent cash flows for the liabilities and eligible replicating assets

2. **optimization** – definition of the objective function and constraints used to solve the optimization problem

3. **analysis** – analysis of the results to measure the robustness of the solution

Once created, replicating portfolios have a wide range of applications. In each case the replicating portfolio is a computationally efficient proxy for the liability portfolio that enables faster turnaround times for other risk reports. Some sample usages include:

Sensitivity Analysis: Since the replicating portfolios consist of simple financial instruments, various sensitivities (greeks) of the replicating portfolios can be easily determined. The sensitivities of the original liability portfolio can be directly estimated from those of the replicating portfolios. This saves considerable time and effort from the perspective of corporate risk management.

Economic Capital Calculation: Consider a large insurance company with multiple business units throughout the globe who wants to compute economic capital. The business units use a wide range of actuarial systems making it difficult to bring all the liability models into a single consistent framework. At the corporate level, a single economic scenario set is generated and sent to each business unit. Each business unit uses its respective models to generate scenario dependent cash flows and transmits the resulting cash flow file back to the corporate entity. The corporate entity then constructs a replicating asset portfolio for each business unit, which is then used as a proxy for each business unit's liability in an economic capital calculation.

Benchmarking Investment Manager Performance: All insurance companies actively monitor the performance of both internal and external investment managers. In this case, the replicating portfolio becomes a target benchmark that can be used to guide investment decisions.

Risk Management: Replicating portfolios can be used as proxies for the actual liability portfolios in calculating various enterprise risk measures, such as Value-at-Risk, Tracking Error or Expected Shortfall.

Solvency II: Using the standard approach or the internal model approach (with the regulator's approval), the replicating portfolios can be used to determine the economic capital charge for market risk.

REPLICATING PORTFOLIO THEORY

The fact that a portfolio that replicates cash flows of a liability today can be used to value the same liability in the future follows from risk-neutrality. In particular, the value of the liability can be written (taking some notational liberties) as:

$$V(t) = E_X\left[\sum_{\tau \in Times} df(\tau, x) cf(\tau, x)\right]$$

where:

E_X *risk - neutral expectation over risk factors X*

$df(t,x)$ *discount factor at time t given risk factors x*

$cf(t,x)$ *cash flow at time t given risk factors x*

A (perfect) replicating portfolio is a set of assets such that the cash flows of the assets match those of the liability:

$$cf(t,x) = \sum_{a \in Assets} x_a cf_a(t,x)$$

where:

x_a *weight of asset a in the replicating portfolio*

$cf_a(t,x)$ *cash flow at time t given risk factors x of asset a*

Given this, the value of the liability at any time point can be written as the value of the assets:

$$V(t) = E_X\left[\sum_{\tau \in Times} df(\tau, x) cf(\tau, x)\right] = E_X\left[\sum_{\tau \in Times} df(\tau, x) \sum_{a \in Assets} x_a cf_a(\tau, x)\right] = \sum_{a \in Assets} x_a V_a(t)$$

Of course, in practice, the cash flows of the replicating portfolio will never equal the cash flows of the assets exactly. The key is to build a replicating portfolio where the cash flows are similar over a wide variety of scenarios (i.e. risk factor realizations). In this regard, optimization is a natural approach to help minimize the mismatches between the asset cash flows and the liability cash flows of an asset portfolio.

STEPS TO BUILD A REPLICATING PORTFOLIO

Algo Risk offers a comprehensive replicating portfolio solution that can be customized to specific client needs. The typical steps to build a replicating portfolio in Algo Risk involve the following steps:

1. Pre-optimization

a. create scenarios on interest rates, equity returns, and other relevant market risk factors over the runoff of the liability

b. generate the cash flows for each liability under each scenario

c. define the tradable universe of assets eligible for inclusion in the replicating portfolio

d. generate the cash flows for each of the assets in the tradable universe under each scenario

2. Optimization

e. specify an objective function that defines how closely asset cash flows match the liability cash flows

f. specify constraints on the optimal portfolio (i.e. no short sales allowed, or no more than 10% allocated to AA bonds, etc)

g. run the optimizer to determine the optimal replicating portfolio

3. Analysis

h. assess how well the replicating portfolio replicates the target liability portfolio

i. test the fit of the replicating portfolio with out of sample scenarios

j. compare the results under different objective functions and constraints

Figure 1 outlines the architecture and data flow in Algo Risk as it applies to the steps required to build a replicating portfolio.

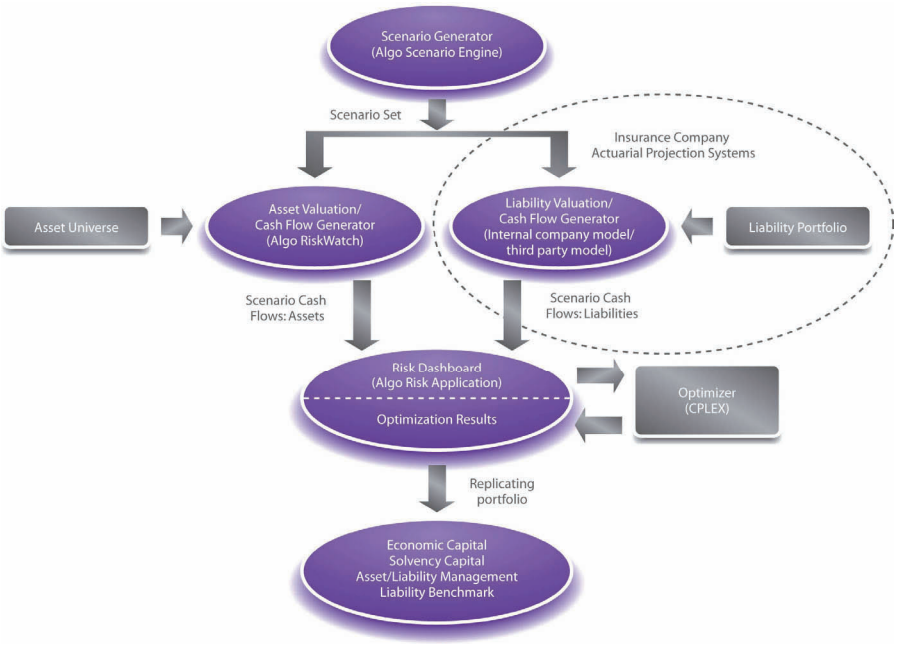

Figure 1 – Algo Risk replicating portfolio architecture

ARCHITECTURE COMPONENTS

Algo Scenario Engine (ASE) & Scenario Generation – Portfolio replication requires replicating cash flows over a set of scenarios. Any set of scenarios can be used as long as it reasonably covers all possible paths that the underlying risk factors can take. Both economic and risk-neutral scenarios may be used. Within Algo Risk, the Algo Scenario Engine (ASE) is used to generate scenarios. ASE supports a wide variety of models for generating economic scenarios calibrated to historical data and risk-neutral scenarios calibrated to current market rates. As an alternative to ASE, users can supply their own scenario sets.

RiskWatch (RW) & Asset Cash Flows – The first step in generating asset cash flows is to determine a suitable universe of replicating assets. The universe typically includes vanilla interest rate products, market indices, and options/forwards on interest products and market indices. RiskWatch uses over 700 asset models to value and settle asset cash flows over time and scenarios. A key feature of the asset models is the ability to accurately simulate over time. An implication is that under a given set of scenarios, options will only be exercised when it is optimal to do so (in a given scenario) so that the asset cash flows are truly scenario dependent. These cash flows are then brought into Algo Risk Application (ARA) for optimization and analysis.

Actuarial Projection System & Liability Cash Flows – After the scenarios are generated, the existing client liability projection systems are used to generate the liability cash flow projections across the scenario set. These cash flows are then brought into Algo Risk Application (ARA) for optimization and analysis. ARA can be used to set up portfolio hierarchies and aggregate/ disaggregate the cash flows based on multiple criteria such as business unit, product type, etc. The choice of the aggregation grouping is determined by the final analysis, such as, are we allocating capital at the level of product type or business unit, or are we hedging a particular guarantee?

Algo Risk Application (ARA) & Optimization – The actual optimization problem is setup in ARA, a web-based tool that allows users to easily create replicating portfolios and incorporate them into enterprise-wide, integrated risk reports. ARA is used to configure how to penalize cash flow mismatches as well to place any constraints on the asset portfolio that may be appropriate. Typical functions for penalizing cash flow mismatches include the absolute cash flow mismatch and squared cash flow mismatch. Other possibilities include expected shortfall measures, maximum errors, and measures that penalize just positive or negative mismatches. In the case where the user chooses to minimize squared cash flow mismatches, the formal expression for the objective function that is being minimized is:

$$min_{x_a} \sum_{t \in Times} w_t \sum_{s \in Scenarios} p^s \left(\sum_{a \in Assets} x_a cash_a(t,s) - cash(t,s) \right)^2$$

where:

w_t *weight on errors at time t*

p^s *probability of scenario s*

The weights and probabilities can be defined by the user, although it is common to set them both to constant values. In addition, although the problem above has been expressed in terms of cash flows at time t, it can equally be written in terms of cash flows discounted to the current time.

Constraints are not always required when building a replicating portfolio, especially if the purpose is to build a portfolio that simply proxies the liability for use in additional calculations. However, when the resulting portfolio is to be used to hedge the liability or guide investment decisions, they are often essential. The optimizer in ARA allows users to place a wide variety of constraints on the asset portfolio, including: restrictions on the size of the asset holdings or trades, restrictions on the composition of the asset portfolios (for example, the value of BBB rated bonds must be less that 10% of total assets), restrictions on the number of assets or trades, and restrictions on the cost of the replicating portfolio.

The user can also include their current asset holdings in the asset universe. In this case, the optimizer will indicate purchases or sales of these assets. The user may restrict both the number and size of these transactions, as well as incorporate transaction costs. After the problem is configured, ARA passes the problem definition and any required data to ILOG CPLEX, an industry standard optimization engine for solving linear, mixed integer, quadratic and mixed integer quadratic programs. CPLEX returns the optimal portfolio to ARA for display and further analysis.

Algo Risk Application (ARA) & Analysis – ARA gives the user the ability to stress the resulting replicating portfolio against both in and out-of-sample scenarios for comprehensive analysis and diagnostics. Using ARA it is easy to perform multiple optimizations using different assumptions, for example different scenario sets, or different constraints, and to compare the results.

REPLICATING PORTFOLIO EXAMPLE IN ALGO RISK

In order to illustrate the capabilities provided within Algo Risk to create and analyze a replicating portfolio, consider the following small example:

Inputs

1. Scenario set: 100 risk neutral scenarios on the US interest rate curve and the S&P 500 projected out 40 years with annual time steps

2. Liability cash flows: the cash flows for a portfolio of liabilities across the 100 scenarios, projected out 40 years with annual time steps

3. Universe of replicating assets: a set of zero coupon bonds, swaptions, market index options (calls and puts), and market index futures with a range of strikes and maturities that span the 40 year project period

4. Asset cash flows: the asset cash flows for each security in the universe across the 100 scenarios, projected out 40 years with annual time steps

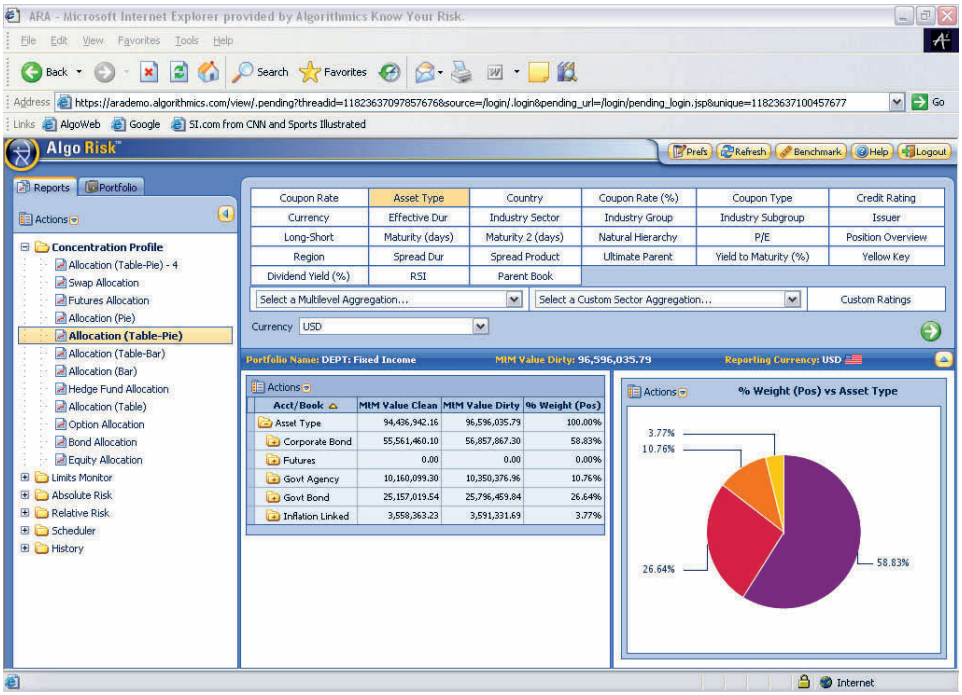

Figure 2 – Algo Risk Application user interface

Optimization Setup: The objective function is defined to minimize the difference in the cash flows between the liability portfolio and the replicating portfolio across the 100 scenarios over the 40 year projection. All the asset weights are constrained to be greater than zero thereby eliminating any short sales in the replicating portfolio.

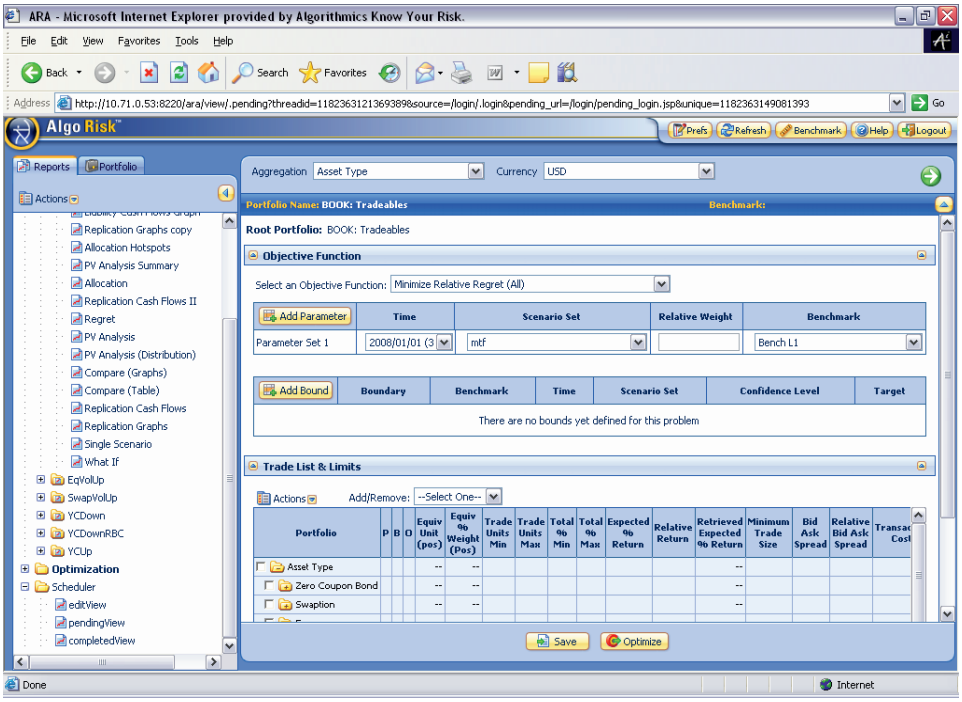

Figure 3 – Optimization setup in ARA

Hot spot analysis: Once the replicating portfolio has been created, ARA can be used to analysis the properties of the replicating portfolio. One example is the makeup of portfolio. In Figure 3, the value of the assets is grouped by asset type and cash flow date, showing that the replicating portfolio contains a high concentration of swaptions with maturities less than ten years.

Actions | Maturity (Years) - Asset Type | Output: Fair Value (Pos)

	European	Equity Forward	Synthetic Instrument	Swaption	Zero Coupon Bond
1 yr	0.0631	0.0000	0.0000	1.0610	0.0000
2 yr	0.0438	0.0000	0.0000	1.4959	0.0000
3 yr	0.0560	0.0000	0.0000	1.0442	0.3502
4 yr	0.0534	0.0000	0.0000	1.0813	0.3395
5 yr	0.0210	0.0000	0.0000	1.2155	0.2839
6 yr	0.0183	0.0000	0.0000	1.3675	0.2725
7 yr	0.0185	0.0000	0.0000	1.2603	0.2238
8 yr	0.0629	0.0000	0.0000	1.2856	0.1490
9 yr	0.0568	0.0000	0.0000	1.0798	0.1444
10 yr	0.0506	0.0000	0.0000	0.9339	0.0857
11 yr	0.0327	0.0000	0.0000	1.1632	0.1419
12 yr	0.0335	0.0000	0.0000	0.8139	0.0854
13 yr	0.0201	0.0000	0.0000	0.8100	0.0655
14 yr	0.0282	0.0000	0.0000	0.5704	0.0175
15 yr	0.0196	0.0000	0.0000	0.5553	0.0236
16 yr	0.0279	0.0000	0.0000	0.5974	0.0317
17 yr	0.0195	0.0000	0.0000	0.2732	0.0081
18 yr	0.0221	0.0000	0.0000	0.3129	0.0145
19 yr	0.0186	0.0000	0.0000	0.2979	0.0327
20 yr	0.0136	0.0000	0.0000	0.2454	0.0198
21 yr	0.0091	0.0000	0.0000	0.2068	0.0216
22 yr	0.0050	0.0000	0.0000	0.1704	0.0384
23 yr	0.0080	0.0000	0.0000	0.1821	0.0080
24 yr	0.0006	0.0000	0.0000	0.1055	0.0150
25 yr	0.0025	0.0000	0.0000	0.1026	0.0114
26 yr	0.0008	0.0000	0.0000	0.0569	0.0094
27 yr	0.0021	0.0000	0.0000	0.0837	0.0218
28 yr	0.0008	0.0000	0.0000	0.0591	0.0077

Figure 4 – Hot spot report for a replicating portfolio

Cash flow analysis: Typically, users will also be interested in how well the portfolio actually replicates the liability cash flows. It is possible to graph the cash flows over time for the portfolio, the liabilities, and the difference between them. These graphs are illustrated in Figure 5, Figure 6, and Figure 7.

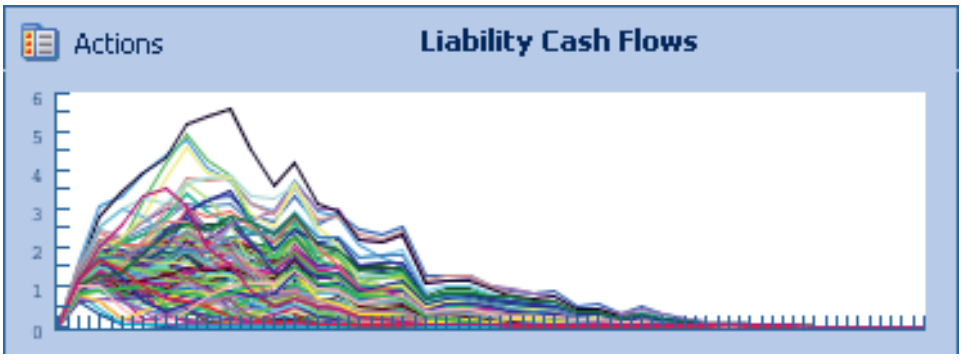

Figure 5 – Liability Cash Flows over 100 scenarios

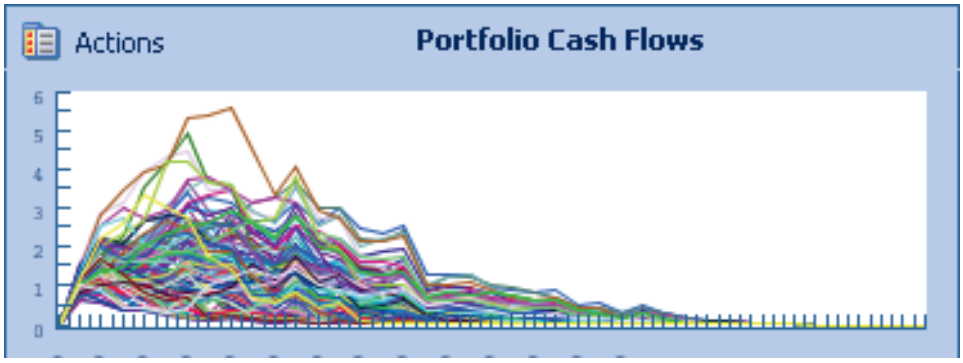

Figure 6 – Portfolio Cash Flows over 100 scenarios

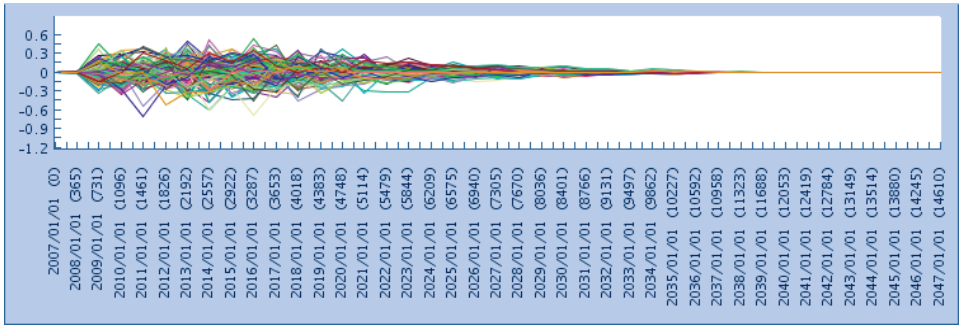

Figure 7: Cash flow mismatches for 100 scenarios

Residual analysis: Another type of report to consider is a residual analysis. This involves comparing the total cash flow mismatch for a scenario against the total cash flow of the liability. Generally speaking one would not want to see any trend in such a report as this would indicate a systemic mismatch. However, in our example a slight downward trend can be seen suggesting that for scenarios where there is very small or very large total liability cash flow there is a greater error.

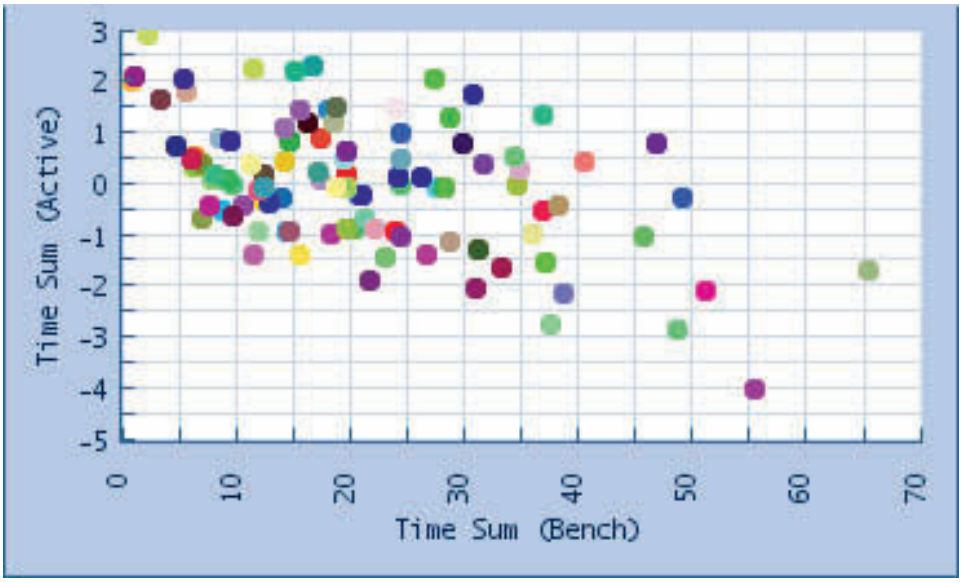

Figure 8 – Residual analysis

Other reports: Once constructed, a replicating portfolio can be analyzed within ARA just as any other portfolio can be analyzed. First off, this means that users can analyze the absolute risk of the liabilities. This analysis, including the liability duration, VaR, and cash flows can be viewed not only with respect to the scenarios used to build the replicating scenarios but any other others also. It is easy to build the replicating portfolio using risk neutral scenarios, for example, but report risk with respect to economic scenarios or stress tests.

Furthermore, not only can the absolute risk of the liabilities be analyzed but, by linking current assets with the replicating portfolio, enterprise-wide risk reports that integrate assets and liabilities are available.

CONCLUSIONS

Algo Risk's replicating portfolio solution provides insurance companies with a way to model liabilities for risk management, capital allocation, and other ALM purposes. Algo Risk's broad asset coverage provides companies with a rich set of potential assets to include in a replicating portfolio. Liability cash flows are generated from existing actuarial projecting systems and integrated with asset cash flows within the Algo Risk Application (ARA).

ARA provides full flexibility in setting up a portfolio replication problem from the choice of the replicating instruments to the ability to define tracking attributes and trading/portfolio constraints. Once a replicating portfolio is created, ARA can be used to analyze the results and then later as the enterprise-wide risk dashboard to create reports on economic capital and capital adequacy, to perform stress tests and what-if analyses, and to generate other advanced ALM reports.

ABOUT THE AUTHORS

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Curt Burmeister is a Senior Financial Engineer at Algorithmics, where he manages the market risk solutions for buy-side clients, including product design, business development, and partner relationships. Having originally joined Algorithmics in 1997, he has worked extensively with clients in North America, Europe, Asia, and Australia to design and implement risk management solutions. Curt holds an M.B.A. in Financial Engineering from MIT, a BA in Computer Science and Mathematics from Cornell University, and has a software patent for complier technology.

Richard Black, Director of Financial Engineering, Algorithmics

Richard Black is a Senior Financial Engineer at Algorithmics, where he manages a team of financial engineers that design and implement analytics in the company's award-winning risk solutions. Prior to joining Algorithmics in 1999, Richard managed the Risk Management Unit at the Bank of Canada after working in the Special Studies and Projection Unit. Richard has a B.Sc. (Honours) in Mathematics from the University of Canterbury, New Zealand, and an M.A. in Economics from Simon Fraser University, British Columbia. His publications have appeared in several economic journals and working paper series.

Appendices:

Proof sources & references

References and success stories from similar Algorithmics projects.

The emphasis should be on what similar problems and challenges were faced by other client organizations and how Algorithmics were able to implement the right solutions to enable these clients to achieve their objectives.

Technical Information

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Contact Information

ABOUT ALGORITHMICS

Algorithmics is the world's leading provider of enterprise risk solutions. Financial organizations from around the world use Algorithmics' software, analytics and advisory services to help them make risk-aware business decisions, maximize shareholder value, and meet regulatory requirements. Supported by a global team of risk experts based in all major financial centers, Algorithmics offers proven, award-winning solutions for market, credit and operational risk, as well as collateral and capital management. Algorithmics is a member of the Fitch Group.

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