Equilibrium

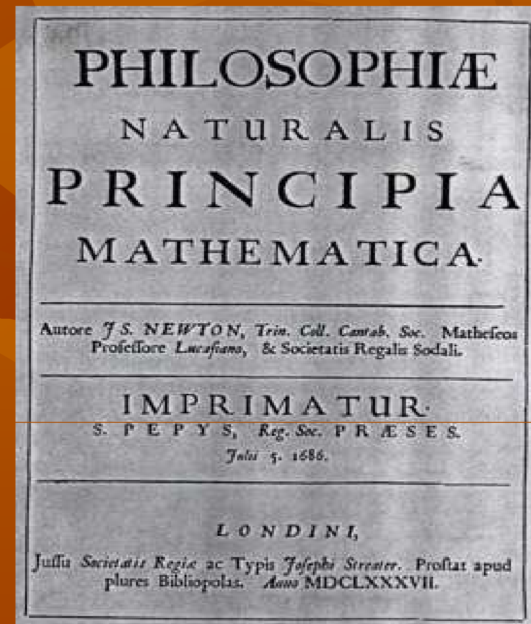
An Educational Software for Engineering Statics

Actus Potentia, Inc.

Contents

(slide numbers in parenthesis)

Isaac Newton



PHILOSOPHIÆ
NATURALIS
PRINCIPIA
MATHEMATICA.

Autore J S. NEWTON, *Trin. Coll. Cantab. Soc.* Matheseos Professore *Lucasiano*, & Societatis Regalis Sodali.

IMPRIMATUR.
S. PEPYS, *Reg. Soc.* PRÆSES.
Julii 5. 1686.

LONDINI,
Jussu *Societatis Regiæ* ac Typis *Josephi Streater*. Prostat apud plures Bibliopolas. *Anno* MDCLXXXVII.

Newton's (1642-1727) systematic and marvelous description of the world in the form of a single set of laws appeared in his book *Philosophiae Naturalis Principia Mathematica* in 1687.

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Newton's First Law

A body remains at rest or, if already in motion, remains in uniform motion with constant speed in a straight line, unless it is acted on by an unbalanced external force.

- Inertia: The *laziness* or *idleness* of a body to maintain the state of motion or the state of rest unless made to do otherwise by an interfering force is called the *Inertia.*

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Explanation of 1st Law

- The inertia of rest is easy to understand and visualize - the car in your garage or the desk in your study will remain immobile forever unless pushed by an external agent.

- The inertia of motion is not readily visible because moving objects around us eventually come to rest under the action of friction from contacting surfaces and/or the aerodynamic drag.

  A hockey puck pushed on a level, cement sidewalk travels along a straight line but comes to a stop due to friction. When the same hockey puck is given the same amount of push on level ice will travel along a straight line much further, because of reduced friction

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Force

- Derived from the *Latin* word *fortis* meaning strong.
- You need to apply a force to push a wheel-barrow across the yard, to throw a basketball to the hoop, or pick up a bucket of water.
- Force is measured in two units: Newton (N) and pounds (lbf).

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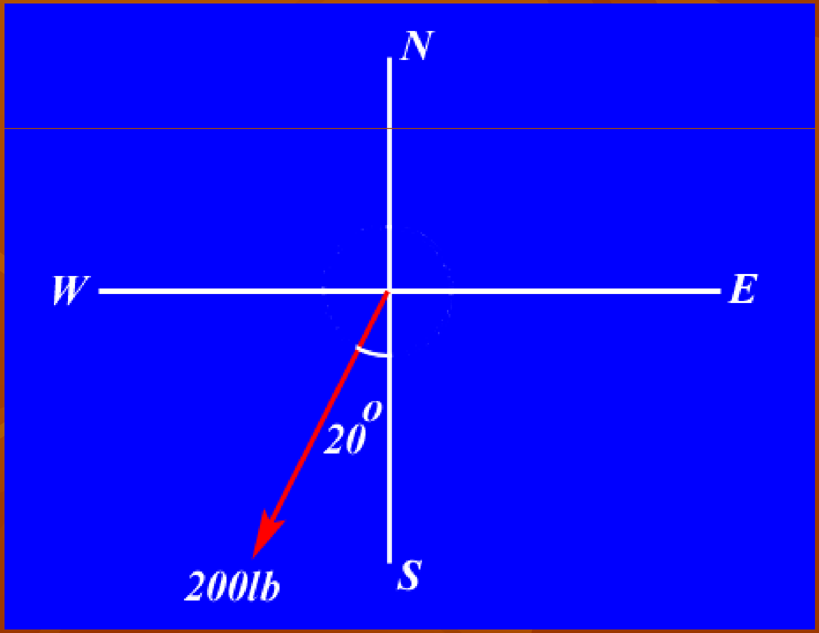
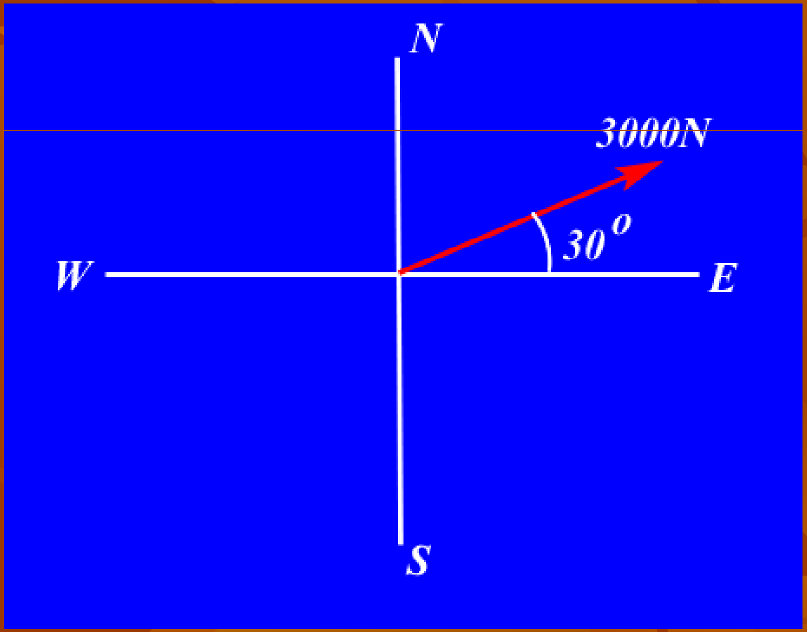
Force is a Vector

Force has magnitude and direction

A 200lb force at an angle of 20^o with South

A 3000N force at an angle of 30^o with East.





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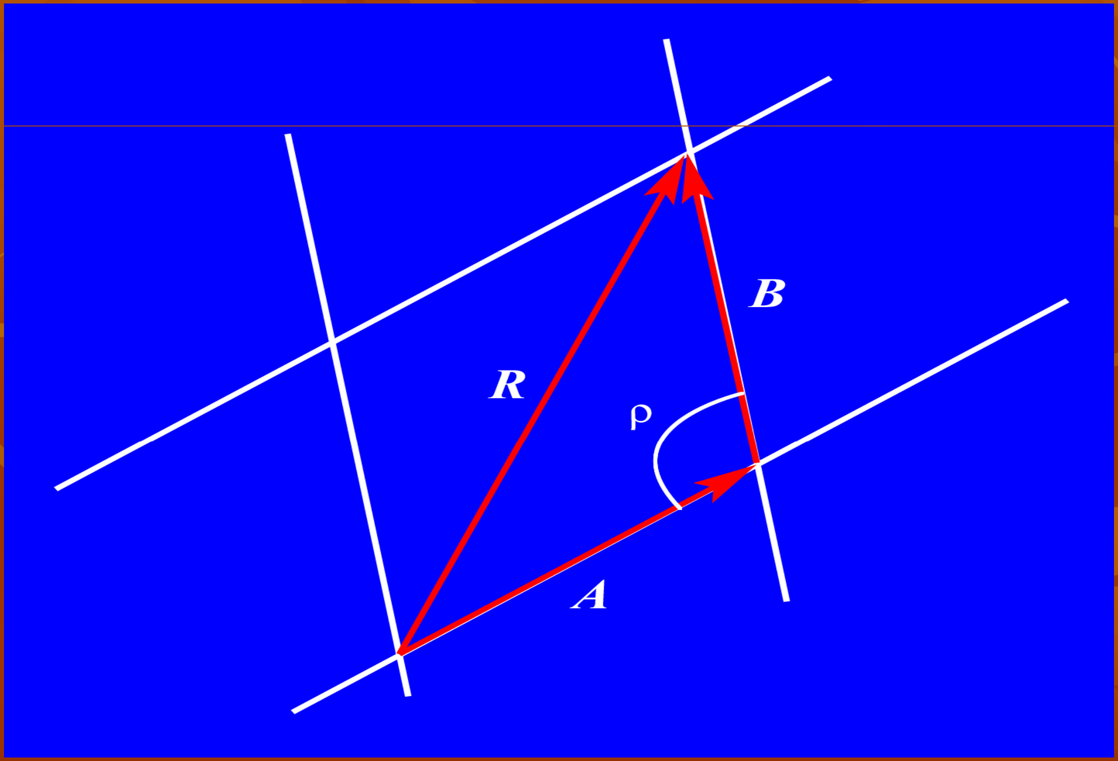
Force is a Vector

Force obeys parallelogram law of addition.

$$\vec{R} = \vec{A} + \vec{B}$$

$$R^2 = A^2 + B^2 - 2AB\cos\rho$$



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Non-rectangular Components of Force

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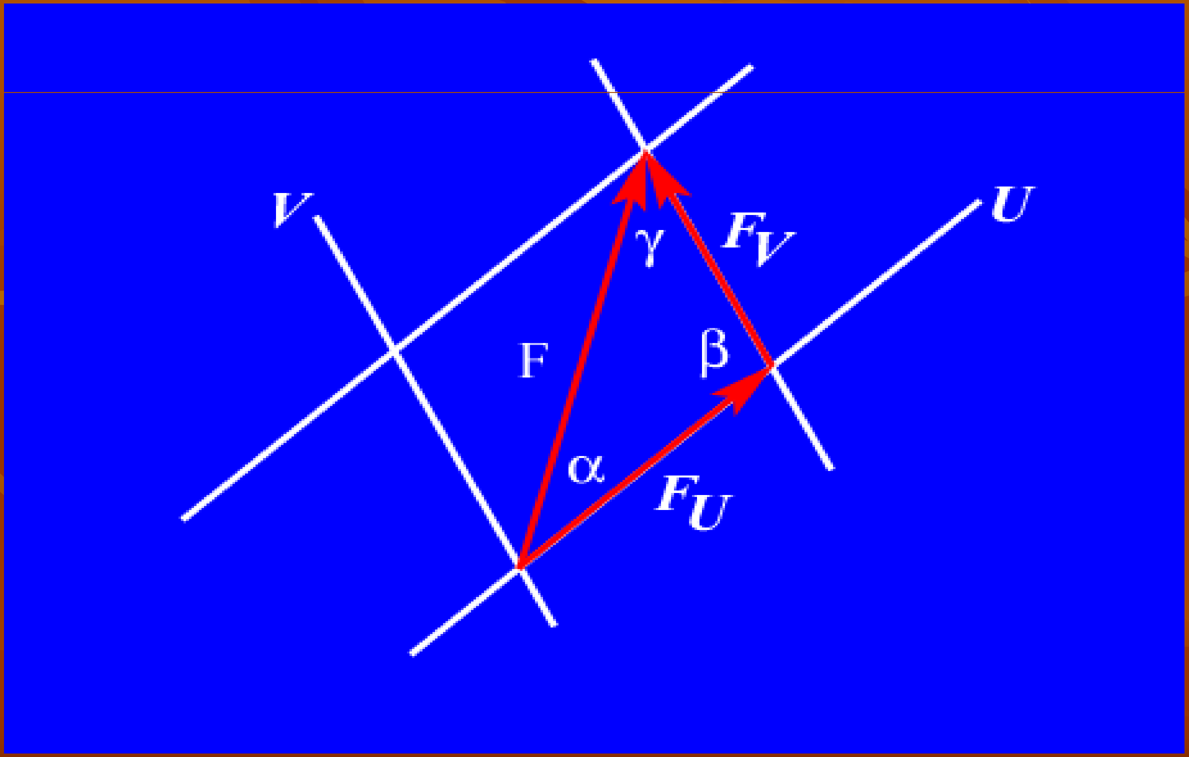
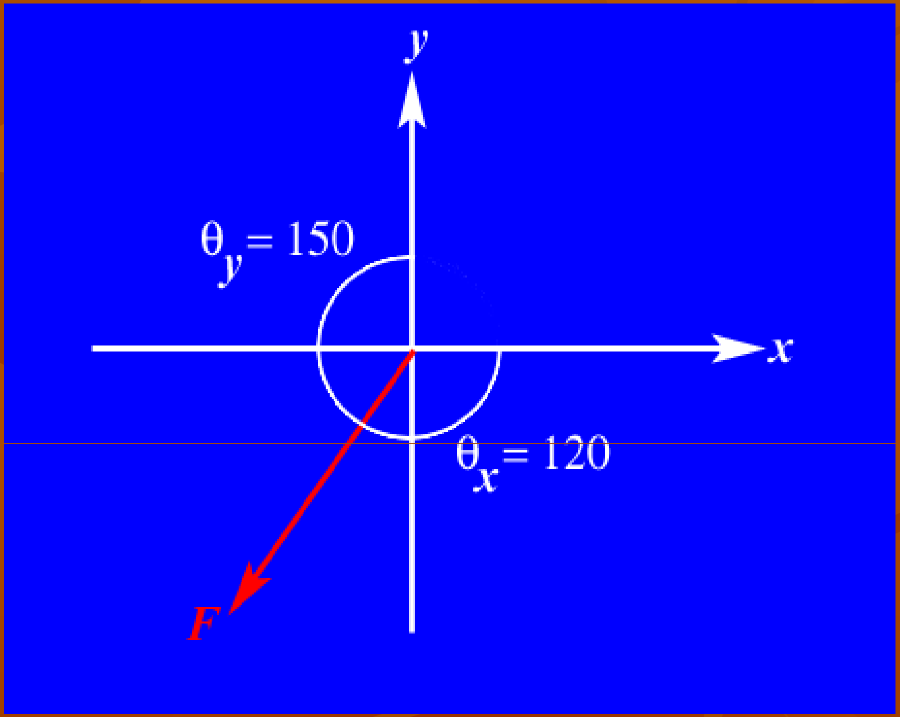
$$\frac{F_U}{\sin\ \gamma} = \frac{F_V}{\sin\ \alpha} = \frac{F}{\sin\ \beta}$$



Rectangular Components of Force

The easiest way to obtain the rectangular components of a force is to use the concept of ***direction cosines***.



The sense of the angle (clockwise/counter-clockwise) between *F* and the axes has no effect, because

$$\cos\theta = \cos(-\theta)$$

$$F_x = F\cos 120^0, F_y = F\cos 150^0$$

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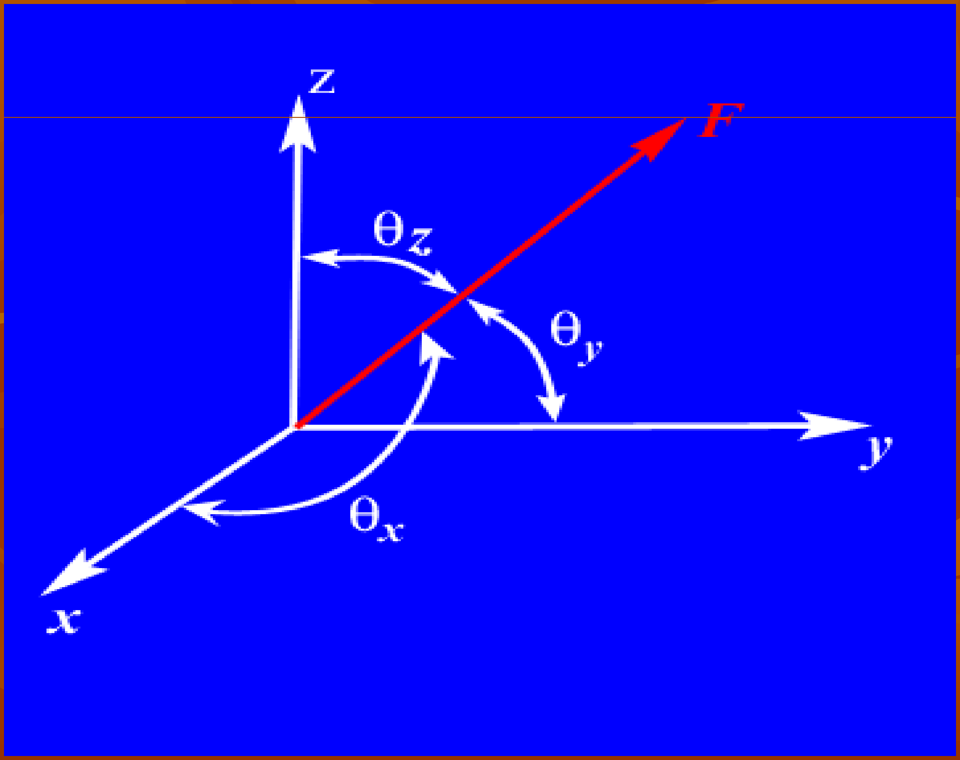
Components in 3-D

$$\vec{F} = F\cos\theta_x \hat{i} + F\cos\theta_y \hat{j} + F\cos\theta_z \hat{k}$$

$$\cos^2\theta_x + \cos^2\theta_y + \cos^2\theta_z = 1$$



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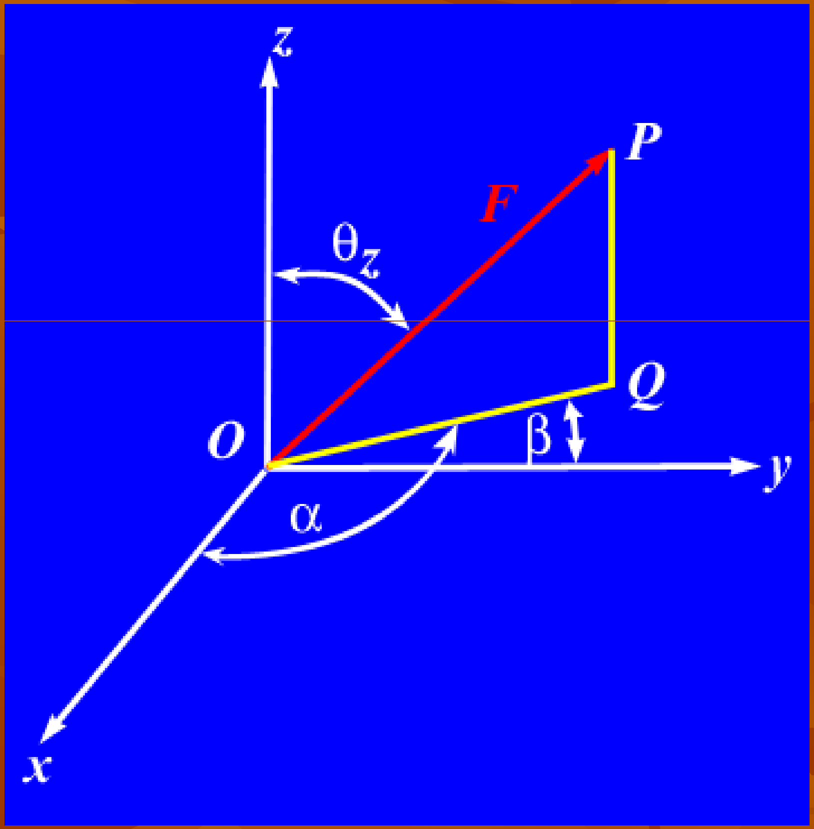
Components in 3-D

Q is the foot of the perpendicular on the xy-plane drawn from the tip P of the force F.

Line OQ is on the xy-plane.



$$F_z = PQ = F\cos\theta_z$$

$$OQ = F\cos(\frac{\pi}{2} - \theta_z)$$

$$F_x = OQ\cos\alpha$$

$$F_y = OQ\cos\beta$$

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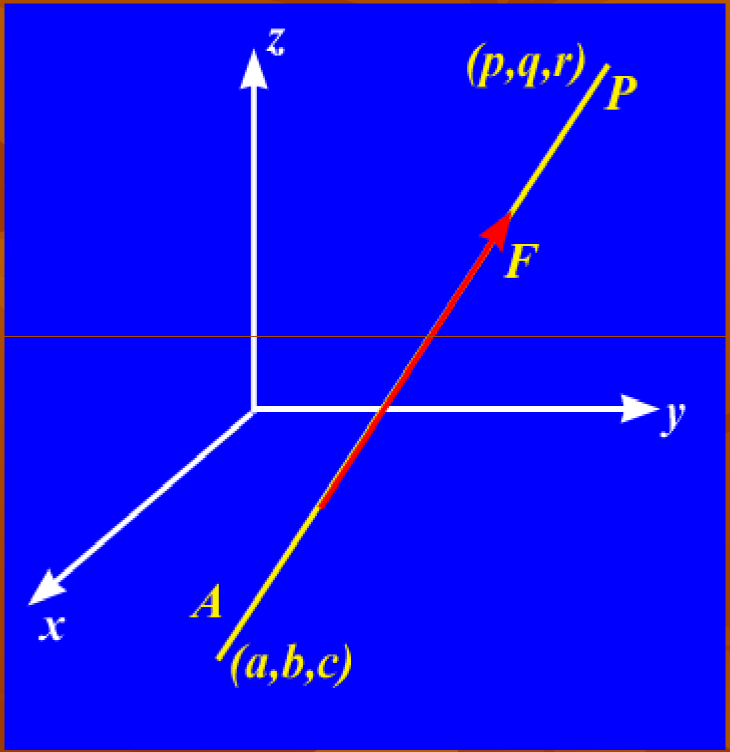
Components in 3-D

A force F is applied along line AP, with its tail towards A and tip towards P.

$$\vec{AP} = (p-a)\hat{i} + (q-b)\hat{j} + (r-c)\hat{k}$$

$$\hat{n}_{AP} = \frac{\vec{AP}}{|\vec{AP}|}$$

$$\vec{F} = F\hat{n}_{AP}$$



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Mass

In 1889, the 1st CGPM (*Conférence Générale des Poids et Mesures*) sanctioned the international prototype of the kilogram. The picture at the right shows the platinum-iridium international prototype, as kept at the International Bureau of Weights and Measures (Sevres, France).

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Newton's Law of Gravitation

Any two bodies in the universe attract each other with a force *(F)* that is directly proportional to the product of the masses (*m and M)* of the bodies, and inversely proportional to the square of the distance (*d*) between them. The constant of proportionality (*G*) is known as the *gravitational constant*.

$$F = \frac{GMm}{d^2}$$

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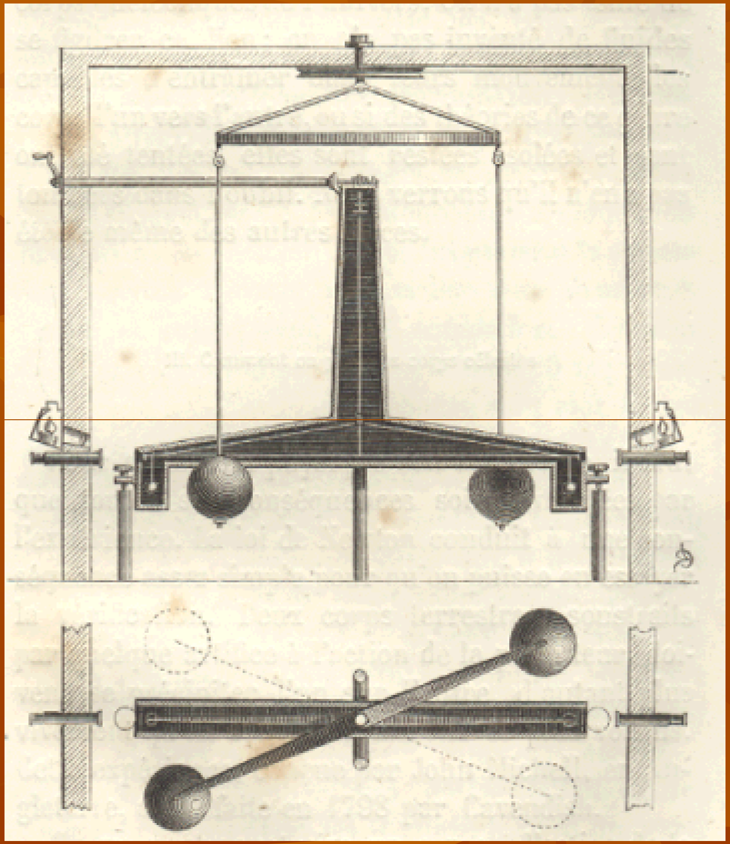
Gravitational Constant

In 1798, Henry Cavendish (1731-1810) used a torsion balance to determine the value of the *gravitational constant (G)*. (see The Feynman Lectures on Physics, Vol-1, p. 7-9, Addison Wesley, 1969).

$$G = 6.670 \times 10^{-11}\, Nm^2 / kg^2$$



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Weight

Weight (*W*) is the gravitational attraction between earth and a body located on the surface of the earth.

Denoting

$$g = \frac{GM}{R^2}$$

$$W = mg$$

M and *R* are mass and radius of earth, respectively.

The quantity *g* is known as the acceleration of gravity.

$$g = 9.81 m/s^2 = 32.2 ft/s^2$$

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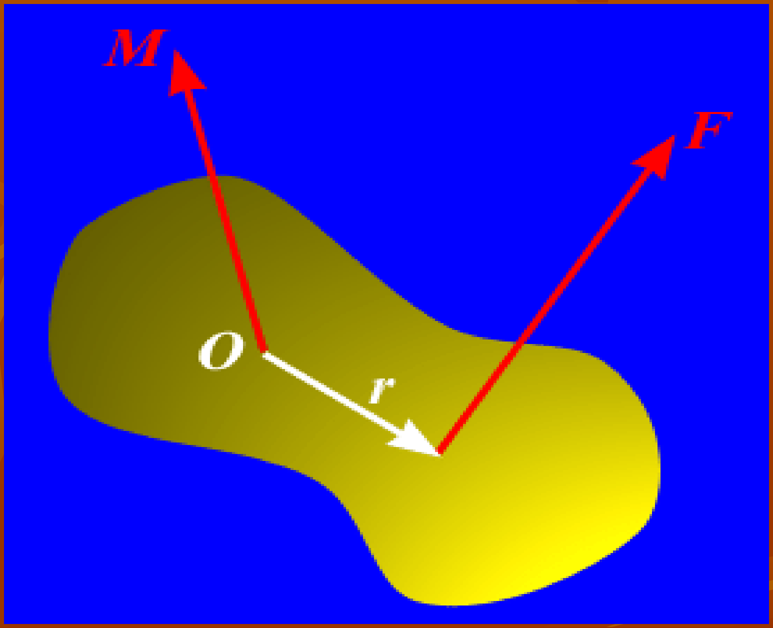
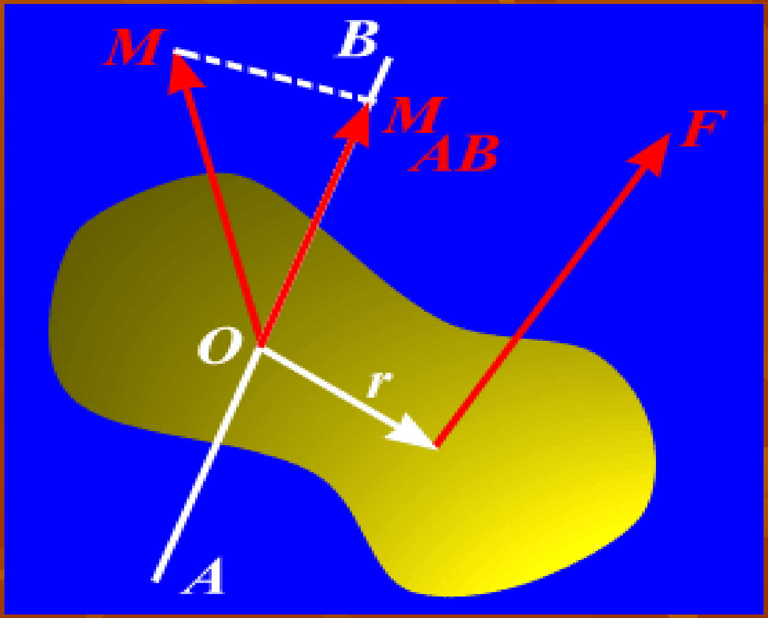
Moment

The moment of a force measures the tendency of a force to make the rigid body rotate about a fixed axis.

Moment can be calculated about a point *O*, or about an axis *AB*.





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Moment as Vector

- Moment about point *O*.

  The tendency of rotation is caused by moment *M*, and the axis of rotation is directed along *M*.

  Vector *r* has its tail at *O* and tip on the line of action of *F*.

$$\vec{M} = \vec{r} \times \vec{F}$$

- Moment about axis *AB*

  The tendency of rotation is caused by moment M_{AB}, and the axis of rotation is directed along $M_{AB.}$

  Vector *r* has its tail on the axis *AB* and the tip on the line of action of F

$$\vec{M}_{AB} = ((\vec{r} \times \vec{F}).\hat{n}_{AB})\hat{n}_{AB}$$

Couple

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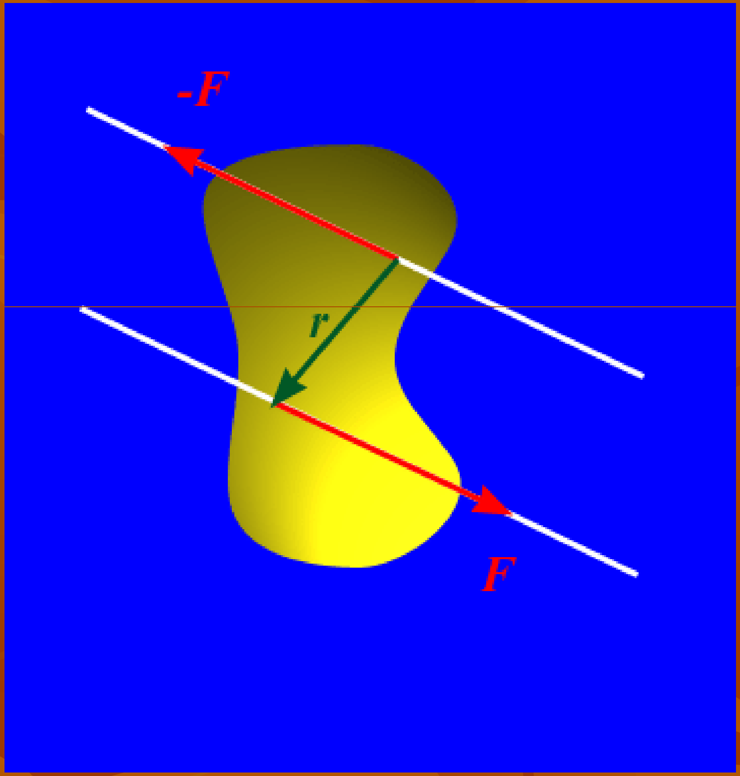
Two equal and opposite forces *F* and –*F* that are separated by a distance form a couple. A couple causes a tendency of pure rotational motion. The moment of the couple is computed from

$$\vec{M} = \vec{r} \times \vec{F}$$

Vector *r* has its tail on the line of action of –*F*, and the tip on the line of action of *F*.



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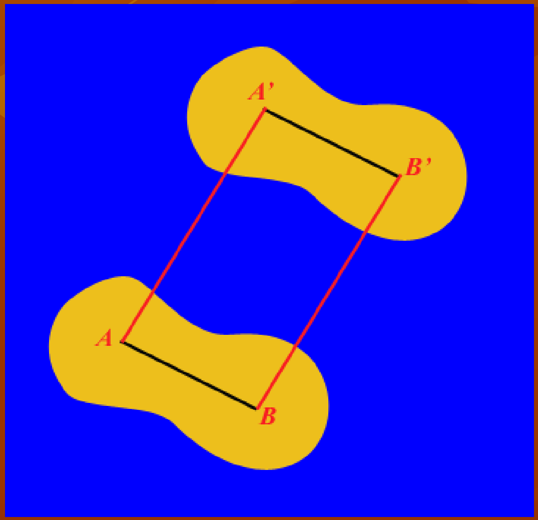
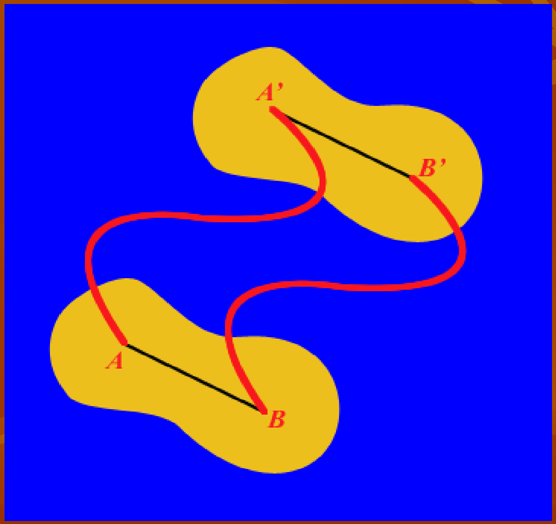
Translational Motion

A a body has translational motion when all the particles forming the body move along parallel paths. Furthermore, a ***line joining two particles (A and B) moves parallel to itself during the motion*.**

If these paths are straight lines, the motion is ***rectilinear translation****;* if the paths are curved lines, the motion is a ***curvilinear translation***.





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Acceleration

- Galileo observed that a body rolling down an inclined plane travels greater and greater distances in successive equal time intervals.
- Galileo quantified the change in velocity with time. This quantification is shown in the Table below. Time is measured in seconds (s), distance in meters (m), and acceleration, in the last column of the Table, in meters/second/second (m/s^2).

Time (s)	Position (m)	Distance Traveled (m)	Average Velocity (m/sec)	Change in Velocity (m/s)	Change in Velocity per second (m/s/s)
0	0				
1	2	2	2		
2	8	6	6	4	4
3	18	10	10	4	4
4	32	14	14	4	4
5	50	18	18	4	4
6	72	22	22	4	4

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Newton's Second Law

Second Law connects the translational acceleration of a body with the resultant force on the body

The acceleration produced by a particular force acting on a body is directly proportional to the magnitude of the force and inversely proportional to the mass of the body.

With acceleration *a,* force *F*, and mass *m,* and constant of proportionality *k*, we get

$$a = \frac{kF}{m}$$

Units

The constant of proportionality *k* = *1* when proper units for acceleration, force, and mass are chosen.

Newton's 2nd Law reduces to

$$\vec{F} = m\vec{a}$$

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Newton's Third law

When a body exerts a force on a second body, the second body exerts a force on the first body. These forces are equal in magnitude, opposite in direction, and have the same line of action.

- (from Understanding Physics, I. Asimov, Vol, 1, George Allen & Unwin Ltd., 1966) People tend to ask: "…if there are two equal and opposite forces, why don't they cancel out by vector addition, leaving no net force at all?" The answer is that two equal and opposite forces cancel out by vector addition when they are exerted on the same body. …The law of interaction, however, involves equal and opposite forces exerted on *two separate bodies*.

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World of Newton

Consider the "World of Newton" that consists of the ***Earth***, ***one apple tree***, and ***one apple***. In this "World of Newton", all the ***celestial and terrestrial bodies are in equilibrium following the same set of laws***.

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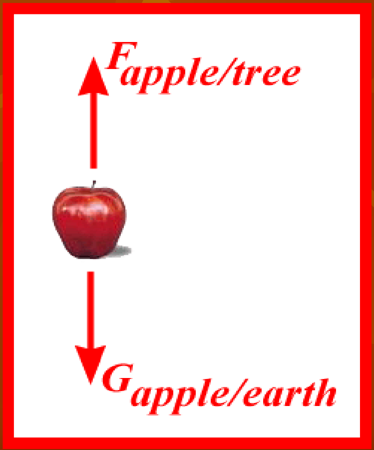
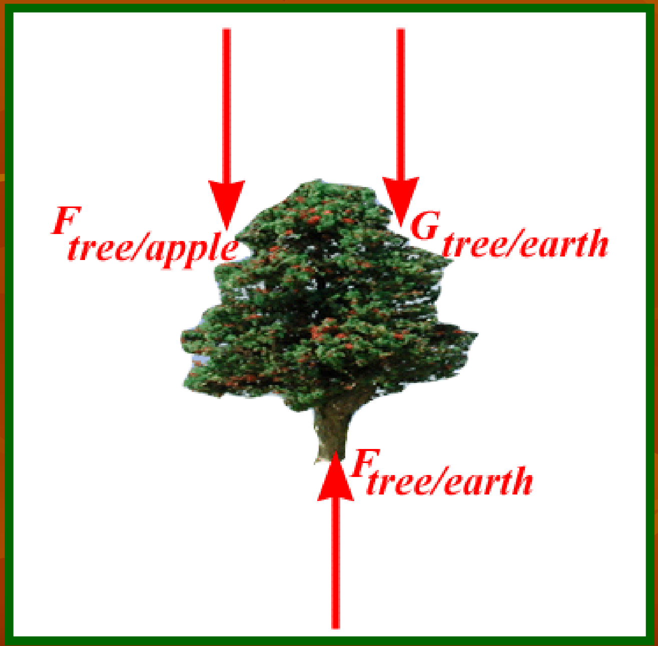
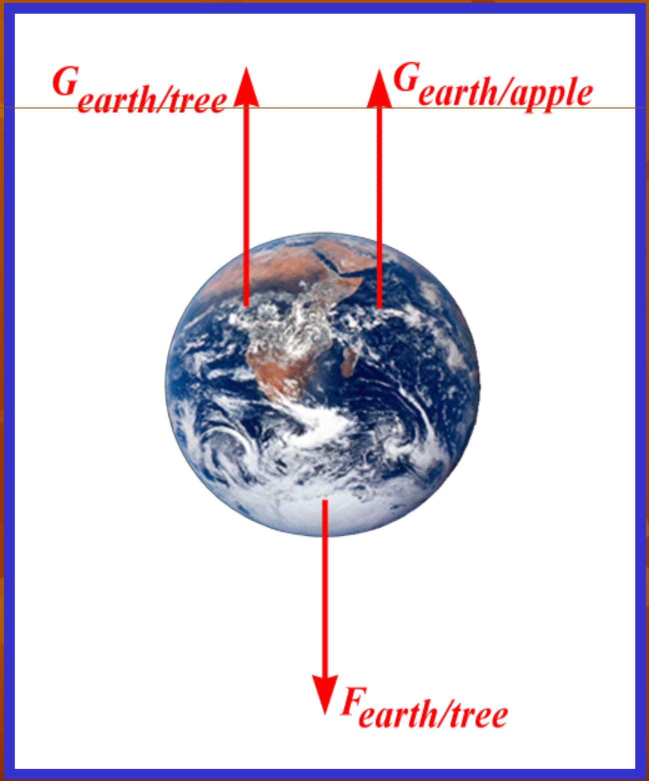
Action-Reaction

In the diagrams, $F_{\alpha/\beta}$ and $F_{\beta/\alpha}$ pairs are equal and opposite, action-reaction, contact forces between objects α and β.

$G_{\alpha/\beta}$ and $G_{\beta/\alpha}$ pairs are equal and opposite, action-reaction, gravitational forces between objects α and β.







Translation Motion of a System(1)

(portions of this material are borrowed from Feynman Lectures on Physics, CA Inst. Of Tech., 1963)

Newton's 2nd Law can easily predict that the flight path of a baseball is parabolic. In this mathematical model, we assume that the baseball is a particle (point mass).

Newton's 2nd Law can also predict the motion of more complicated objects: gas molecules swarming, water flowing, and galaxies whirling.

Consider a complex object consisting of springs, cables, spokes, solid blocks, and buckets of liquid. When this composite object is tossed like a baseball, each individual part may tumble, jiggle, slosh, and vibrate, but "something" in this object continues to travel on a parabolic path. That "something" is called the ***center of mass***.

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Translational Motion of a System(2)

Consider a system of two masses m_1 and m_2 located at r_1 and r_2 in a coordinate system fixed in space. External forces F_1 and F_2 are acting on these two masses. The forces f_{12} and f_{21} are interactive forces among the masses. From Newton's 3rd Law

$$\vec{f}_{12} = -\vec{f}_{21}$$

The equations of motion are

$$\vec{F}_1 + \vec{f}_{12} = m_1 \frac{d^2\vec{r}_1}{dt^2} = \frac{d^2(m_1\vec{r}_1)}{dt^2}$$

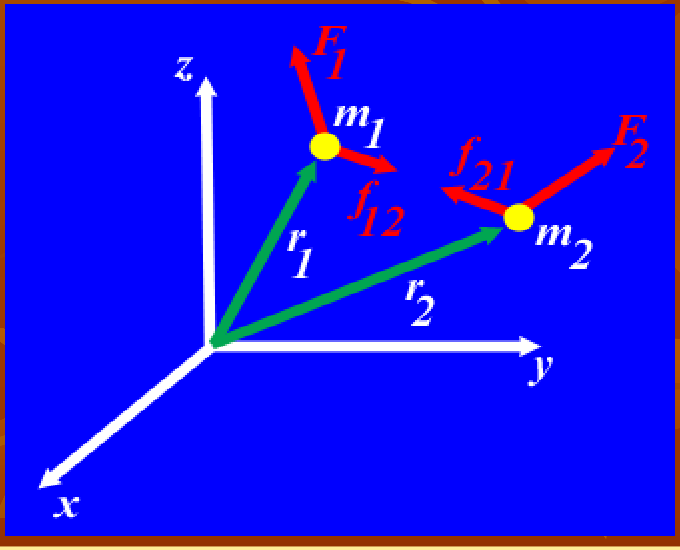
$$\vec{F}_2 + \vec{f}_{21} = m_2 \frac{d^2\vec{r}_2}{dt^2} = \frac{d^2(m_2\vec{r}_2)}{dt^2}$$

Adding the equations of motion

$$\vec{F}_1 + \vec{F}_2 = \frac{d^2(m_1\vec{r}_1 + m_2\vec{r}_2)}{dt^2}$$

For a system of n masses

$$\sum_{i=1}^{n} \vec{F}_i = \frac{d^2(\sum_{i=1}^{n} m_i\vec{r}_i)}{dt^2}$$



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Center of Mass

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From previous slide,

$$\sum_i \vec{F}_i = \frac{d^2(\sum_i m_i \vec{r}_i)}{dt^2}$$

Defining

$$m = \sum_i m_i$$

$$\vec{G} = \frac{1}{m}\sum_i m_i \vec{r}_i$$

where ***G*** is the location of the center of mass,

the equation of motion becomes

$$\sum_i \vec{F}_i = m\frac{d^2\vec{G}}{dt^2}$$

For a continuous body

$$\vec{G} = \frac{\int \rho \vec{r}\, dV}{\int \rho\, dV}$$

Centroid – 2D

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For a slab of uniform density, the location of the centroid is

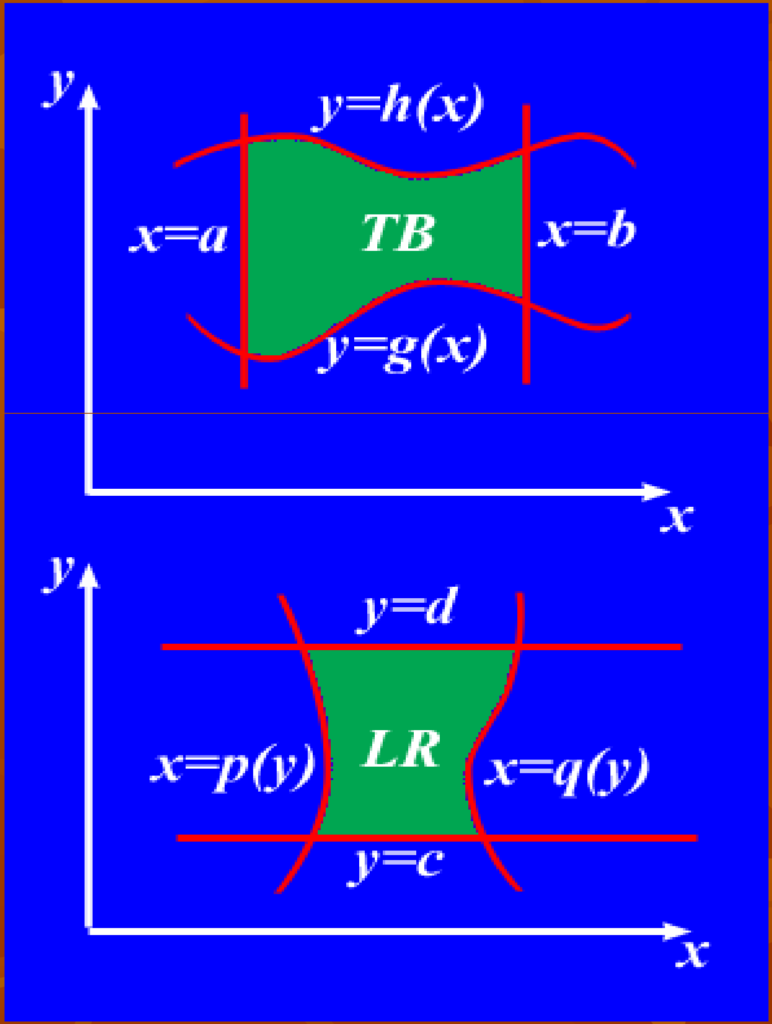
$$\vec{G} = (\bar{x}, \bar{y}) = \frac{\rho \int \vec{r}\, dA}{\rho \int dA} = \frac{\int \vec{r}\, dxdy}{\int dxdy}$$

The integrals are evaluated as ***repeated*** or ***iterated*** integrals. For a ***TB-Region,*** we write

$$\int f(x,y)dxdy = \int_a^b \{ \int_{g(x)}^{h(x)} f(x,y)dy \} dx$$

For a ***LR-Region,*** we write

$$\int f(x,y)dxdy = \int_c^d \{ \int_{p(y)}^{q(y)} f(x,y)dx \} dy$$



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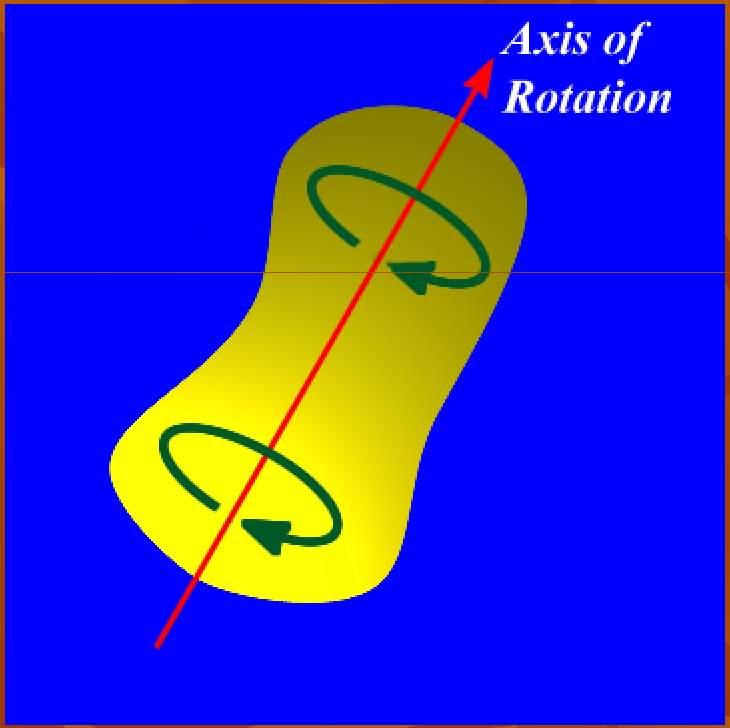
Rotational Motion

A body has rotational motion when all the particles forming the body move on parallel planes along circles centered on the same fixed axis. This axis is called the axis of rotation.



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Rotational Motion of a System

For a system of two masses, we have shown

$$\vec{F}_1 + \vec{f}_{12} = m_1 \frac{d^2\vec{r}_1}{dt^2} = m_1\vec{a}_1$$

$$\vec{F}_2 + \vec{f}_{21} = m_2 \frac{d^2\vec{r}_2}{dt^2} = m_2\vec{a}_2$$

By taking a cross product

$$\vec{r}_1 \times \vec{F}_1 + \vec{r}_1 \times \vec{f}_{12} = \vec{r}_1 \times m_1\vec{a}_1$$

$$\vec{r}_2 \times \vec{F}_2 + \vec{r}_2 \times \vec{f}_{21} = \vec{r}_2 \times m_2\vec{a}_2$$

Also

$$\vec{r}_1\times\vec{f}_{12}+\vec{r}_2\times\vec{f}_{21}=\vec{r}_1\times\vec{f}_{12}+(\vec{r}_1+\vec{r}_{12})\times\vec{f}_{21}$$

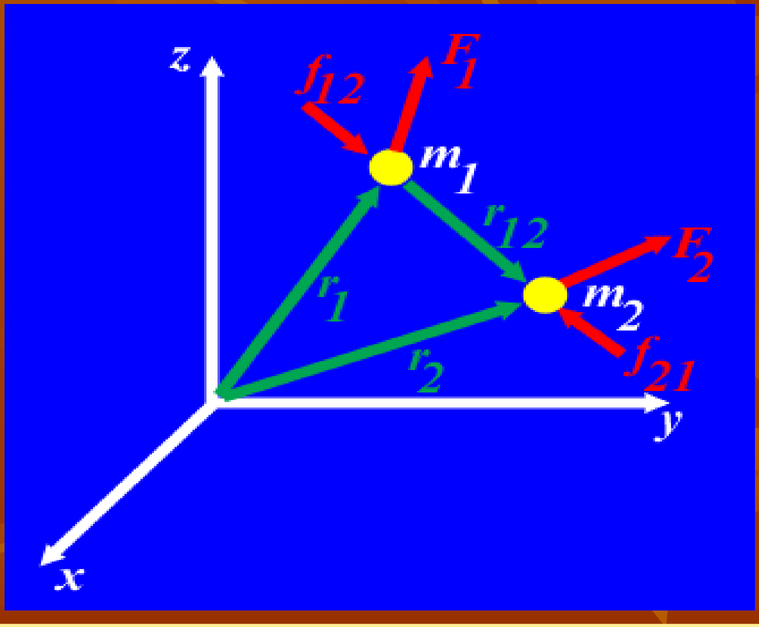
$$=\vec{r}_1\times(\vec{f}_{12}+\vec{f}_{21})+\vec{r}_{12}\times\vec{f}_{21}=0$$

Therefore

$$\sum_{i=1}^{2}(\vec{r}_i \times \vec{F}_i) = \sum_{i=1}^{2}(\vec{r}_i \times m_i\vec{a}_i)$$

For *n* number of masses

$$\sum_{i=1}^{n}(\vec{r}_i \times \vec{F}_i) = \sum_{i=1}^{n}(\vec{r}_i \times m_i\vec{a}_i)$$



Moment and Angular Momentum

When the velocities of the masses in a system are $v_1, v_2,\ldots, v_i,\ldots, v_n$, we find

$$\frac{d}{dt}(\vec{r}_i \times m_i\vec{v}_i) = \frac{d\vec{r}_i}{dt} \times m_i\vec{v}_i + \vec{r}_i \times m_i \frac{d\vec{v}_i}{dt}$$

$$= \vec{v}_i \times m_i\vec{v}_i + \vec{r}_i \times m_i\vec{a}_i = \vec{r}_i \times m_i\vec{a}_i$$

Therefore, the equation of rotational motion of a system becomes

$$\sum_{i=1}^{n}(\vec{r}_i \times \vec{F}_i) = \sum_{i=1}^{n}\frac{d}{dt}(\vec{r}_i \times m_i\vec{v}_i)$$

Or

$$\vec{M} = \sum_{i=1}^{n}\frac{d}{dt}\vec{L}_i = \frac{d\vec{L}}{dt}$$

Where M is the net moment and L is the net angular momentum of the system

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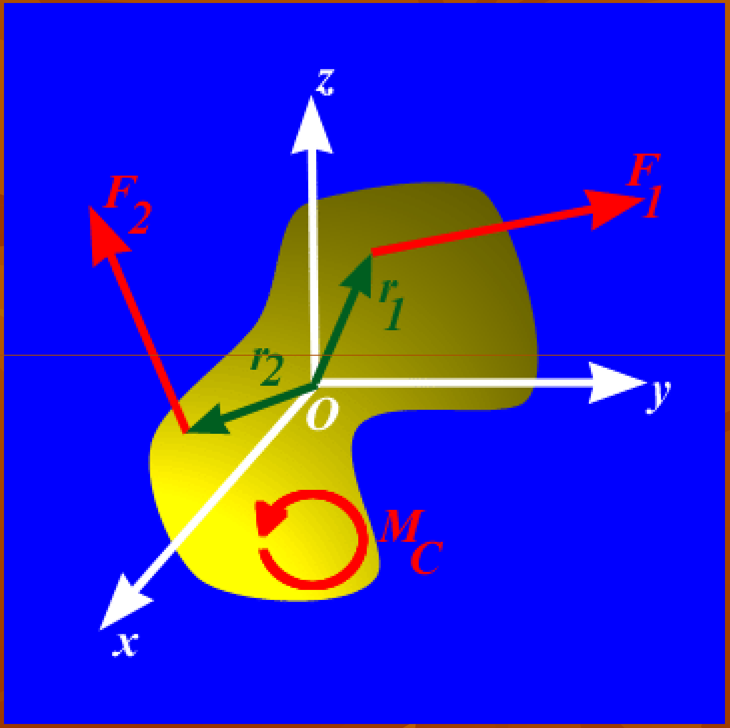
Generalized Equations of Motion

Translational plus Rotational Motion

Two forces F_1, F_2, and a couple M_C are acting on a body. The equations of motion of the body are

$$\vec{R} = \vec{F}_1 + \vec{F}_2 = m\vec{a}_G$$

$$\vec{M}_O = \vec{r}_1 \times \vec{F}_1 + \vec{r}_2 \times \vec{F}_2 + \vec{M}_C = \frac{d\vec{L}}{dt}$$



Where a_G is the acceleration of the center of mass and L is the angular momentum of the body.

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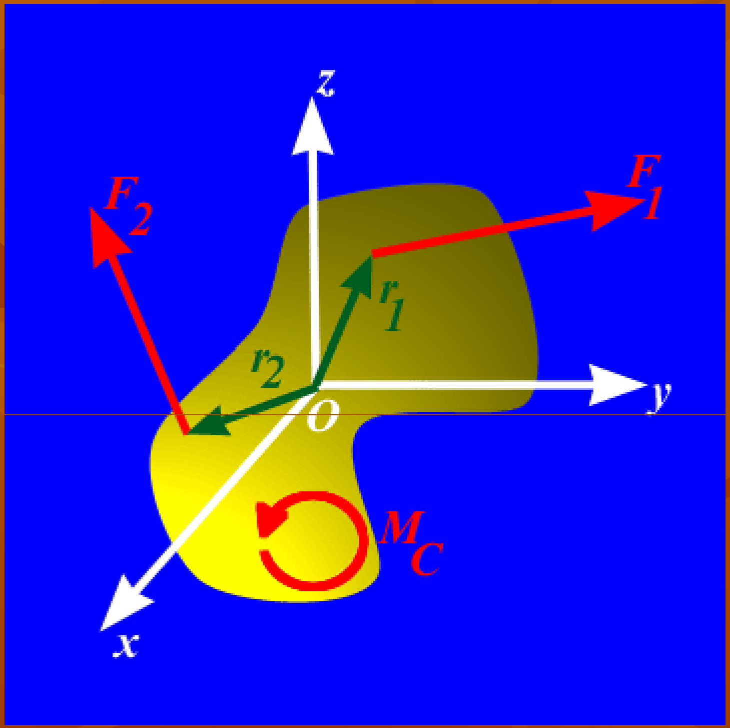
Equilibrium

At equilibrium

$$\vec{a}_G = \frac{d\vec{L}}{dt} = 0$$

Then the equations for force-balance and moment-balance are

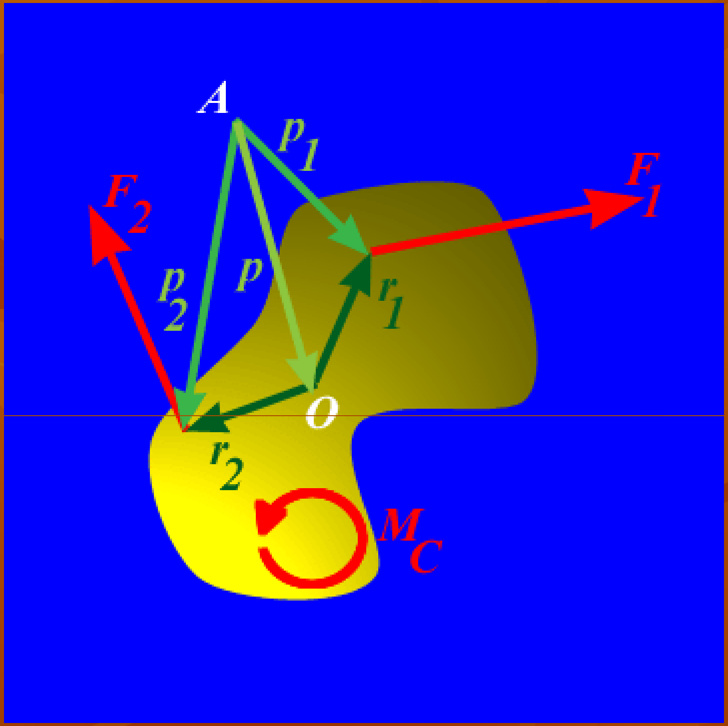


$$\vec{F}_1 + \vec{F}_2 = 0$$

$$\vec{M}_O = \vec{r}_1 \times \vec{F}_1 + \vec{r}_2 \times \vec{F}_2 + \vec{M}_C = 0$$

You can Balance Moment at *any point of your choice*

You need not enforce the moment-balance equation at the origin *O*. ***You can pick any suitable point, including points outside the body***, such as *A*, in the Figure.



$$\vec{M}_A = \vec{p}_1 \times \vec{F}_1 + \vec{p}_2 \times \vec{F}_2 + \vec{M}_C$$

$$= (\vec{p} + \vec{r}_1) \times \vec{F}_1 + (\vec{p} + \vec{r}_2) \times \vec{F}_1 + \vec{M}_C$$

$$= \vec{p} \times (\vec{F}_1 + \vec{F}_2) + \vec{r}_1 \times \vec{F}_1 + \vec{r}_2 \times \vec{F}_2 + \vec{M}_C$$

$$= 0$$

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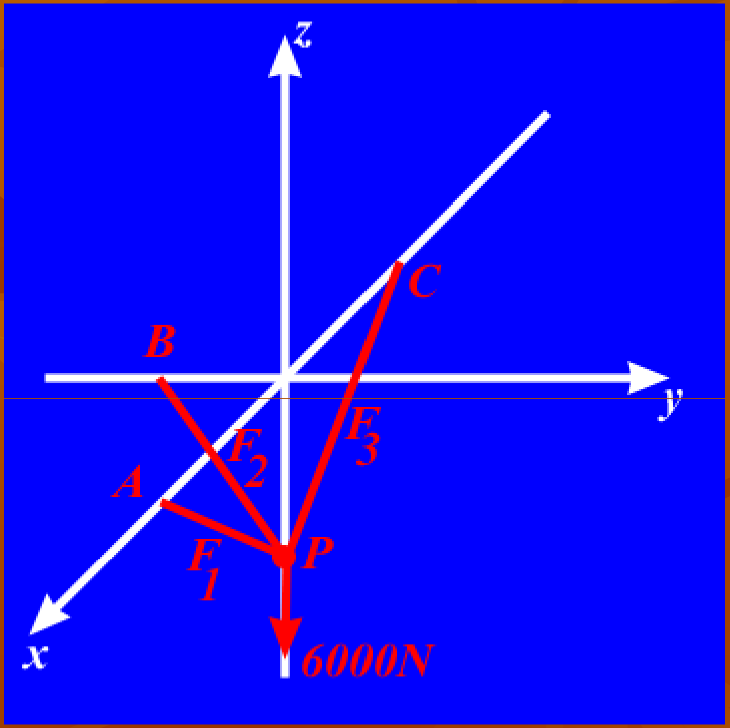
Concurrent Forces
(Equilibrium of a particle)

A 6000N force is applied on a particle. The particle is maintained in equilibrium with 3 cables *PA*, *PB*, and *PC*.



The unit vectors = $\hat{n}$

Equation for equilibrium:

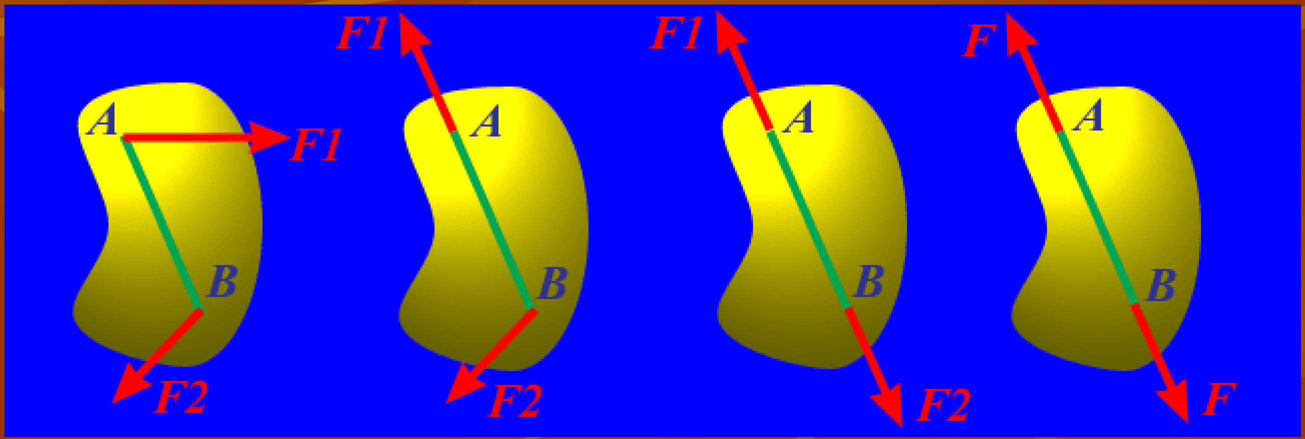
$$F_1\hat{n}_{PA} + F_2\hat{n}_{PB} + F_3\hat{n}_{PC} - 6000\,\hat{k} = 0$$

Equilibrium of Two-Force Members

Consider the diagrams below from left to right, where two forces are acting on a body.

1. Force *F1* and *F2* are acting at points *A* and *B*, respectively.
2. We balance moment at *B*. Moment due to *F2* is zero. For moment due to *F1* to be zero, it must act along line *AB*.
3. We balance moment at *A*. Moment due to *F1* is zero. For moment due to *F2* to be zero, it must act along line *AB*.
4. For two forces acting along line *AB*, we must have *F1* = *F2* = *F*

When two forces act on a body in equilibrium, those two forces must have the same line of action, same magnitude, and opposite sense.



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Friction (1)

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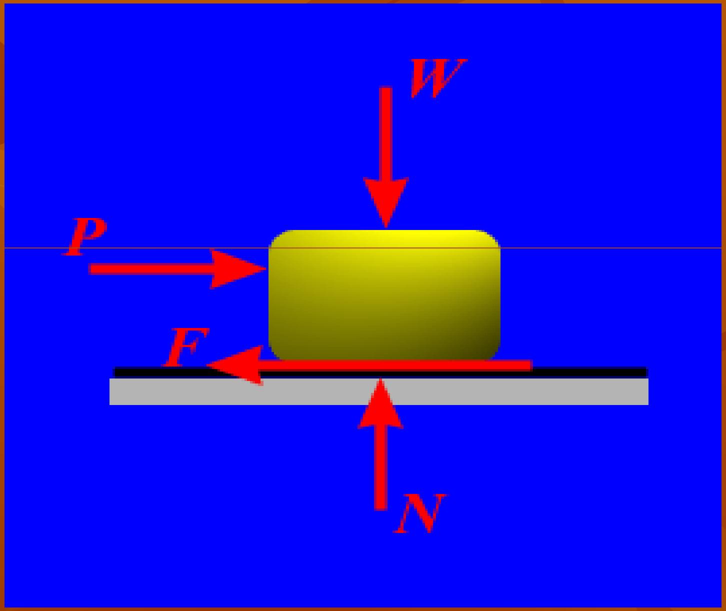
- Friction force is present only when there is motion (kinetic friction) or tendency of motion (static friction).
- The friction force acts opposite to the direction of motion or tendency of motion.
- Friction force acts along the tangent to the contact between two surfaces.
- The roughness of the contact is characterized by the "friction factor", μ.

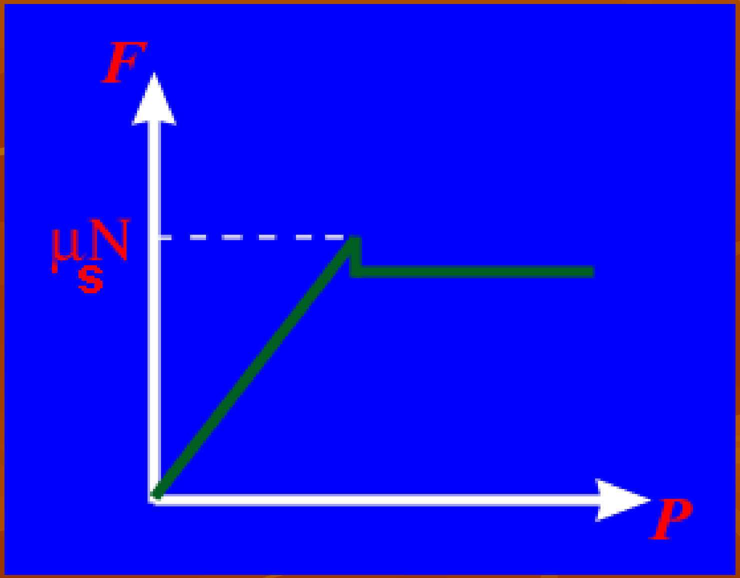


Friction (2)

- Magnitude of friction force when the body is not in motion (static): $F_{stat} \leq \mu_s N$
- The equality holds for impending (about to occur) motion. At this condition, friction force is maximum: $F_{\max} = \mu_s N$
- Magnitude of friction force when the body is in motion (kinetic): $F_{kin} = \mu_k N$

Friction (3)

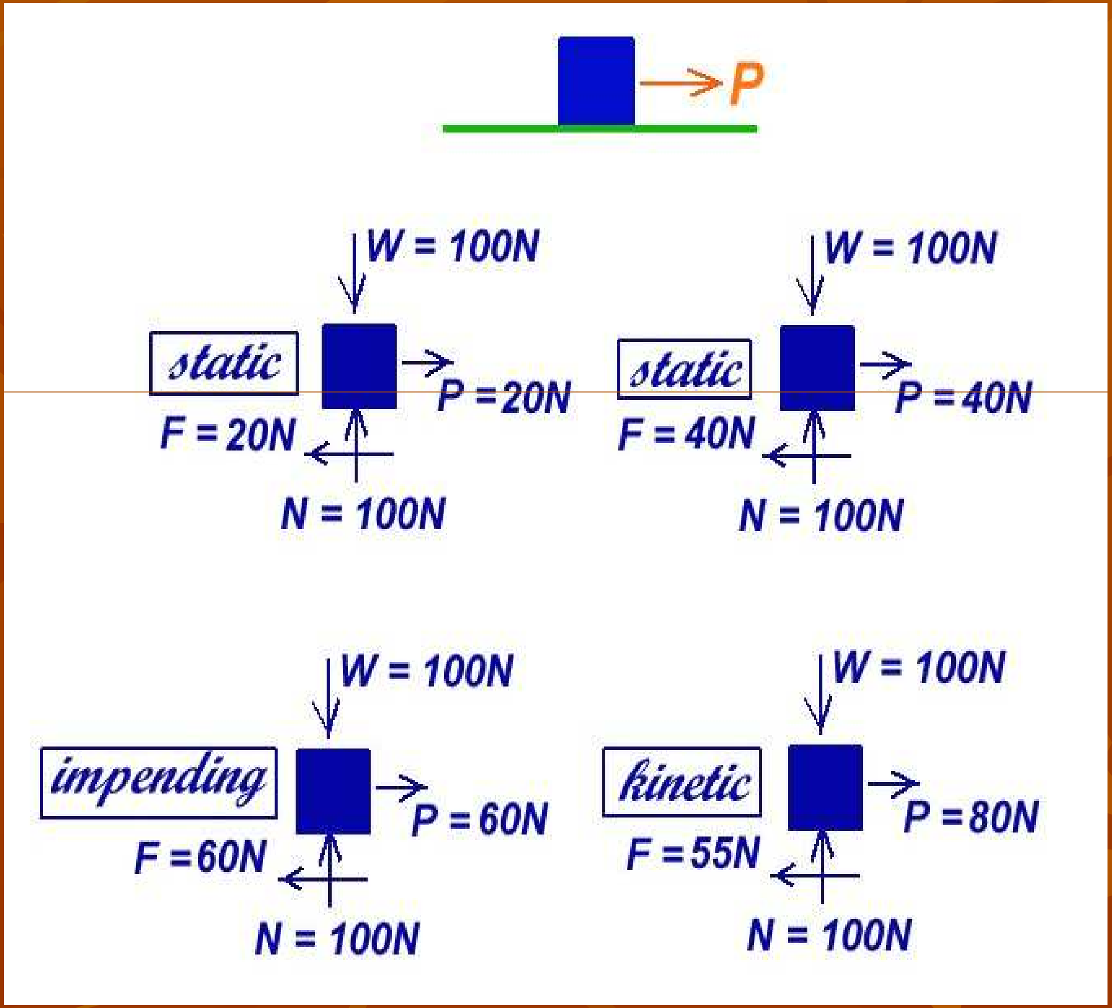
When $P=0$, friction F is also zero. As P increases, F increases to establish equilibrium ($F=P$). The maximum value of F is $\mu_s N$, where μ_s is the coefficient of friction. At $P=\mu_s N$, motion of the body is impending.



Friction (4)

A *100N* body is on a rough floor. Force *P* is applied on the body. The friction coefficient at the contact

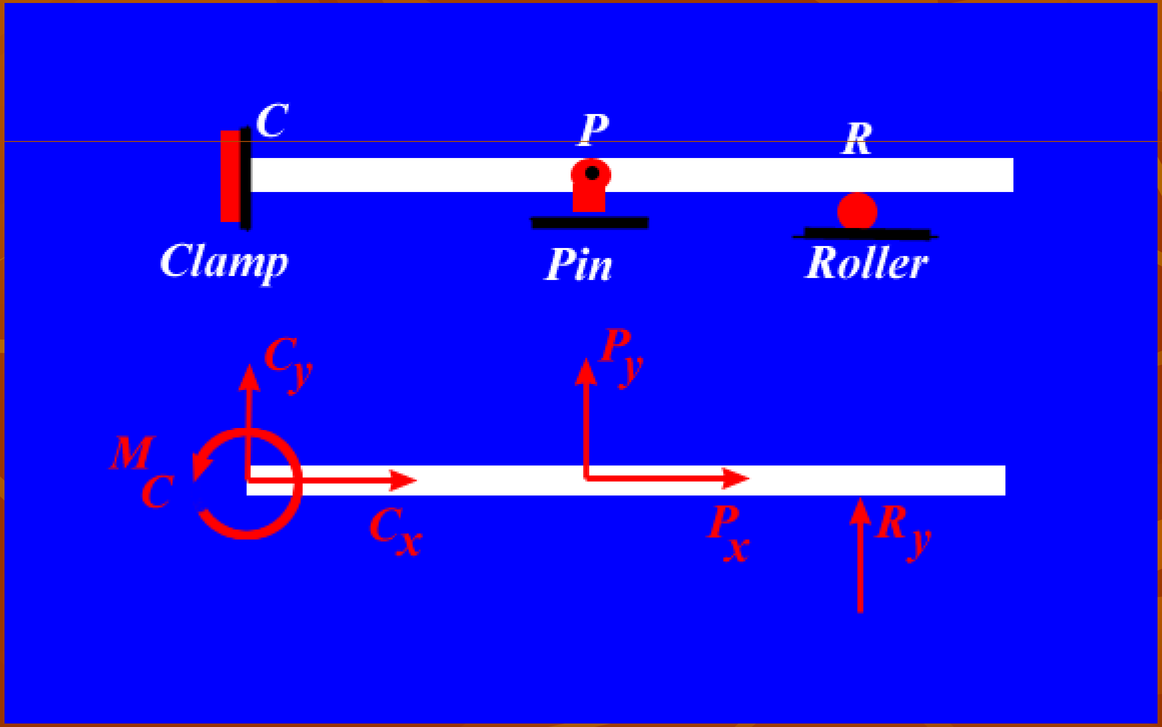
$$\mu_s = 0.6$$
$$\mu_k = 0.55$$



Pin-Roller-Clamp

A body can be supported in 3 different ways – clamp, pin, roller.

The support reactions from these 3 elements are shown in Figure below.



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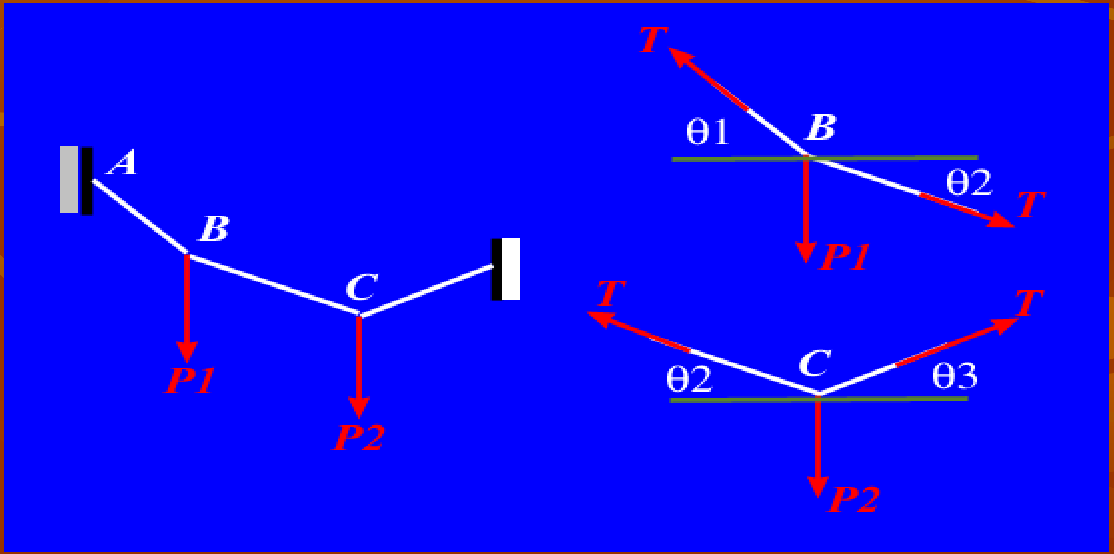
Cables

Cables are considered as weightless components in a structure. Cables are flexible, i.e., its resistance to bending is negligible. The tension ***T*** in the cable remains constant along the length of the cable.

In the loaded cable, there are 4 unknowns: tension ***T***, and the angles θ1, θ2, θ3.

These can be determined by enforcing the equilibrium of points ***B*** and ***C*.**



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Free-Body-Diagram (1)

Contents

- Free-body-diagram (FBD) is a pictorial description of a problem in mechanics.

- One can draw an FBD for a body, or a slice of a body, or a part of a structure consisting of several bodies.

- The body is free because it is isolated from its surroundings by removing it from its supports, by cutting all the cables, and by separating it from other bodies of contact.

- The forces on the body from its surroundings are drawn in the diagram by utilizing Newton's 3rd Law.

- Once this diagram is drawn, Newton's 2nd Law is written for each free-body.

- For a system of bodies, one obtains a system of equations that are solved to obtain the net loading on each free-body.

Free-Body-Diagram (2)

Contents

Steps in drawing an FBD:

1. Set-up a coordinate system
2. Isolate the body of interest
3. Draw the loading that includes: weight of body, applied forces and moments.
4. Draw forces from supports that include: forces from pins, rollers, clamps, and cables.
5. Draw the forces from other bodies that are in contact with the free-body. These contact forces obey Newton's 3rd Law.

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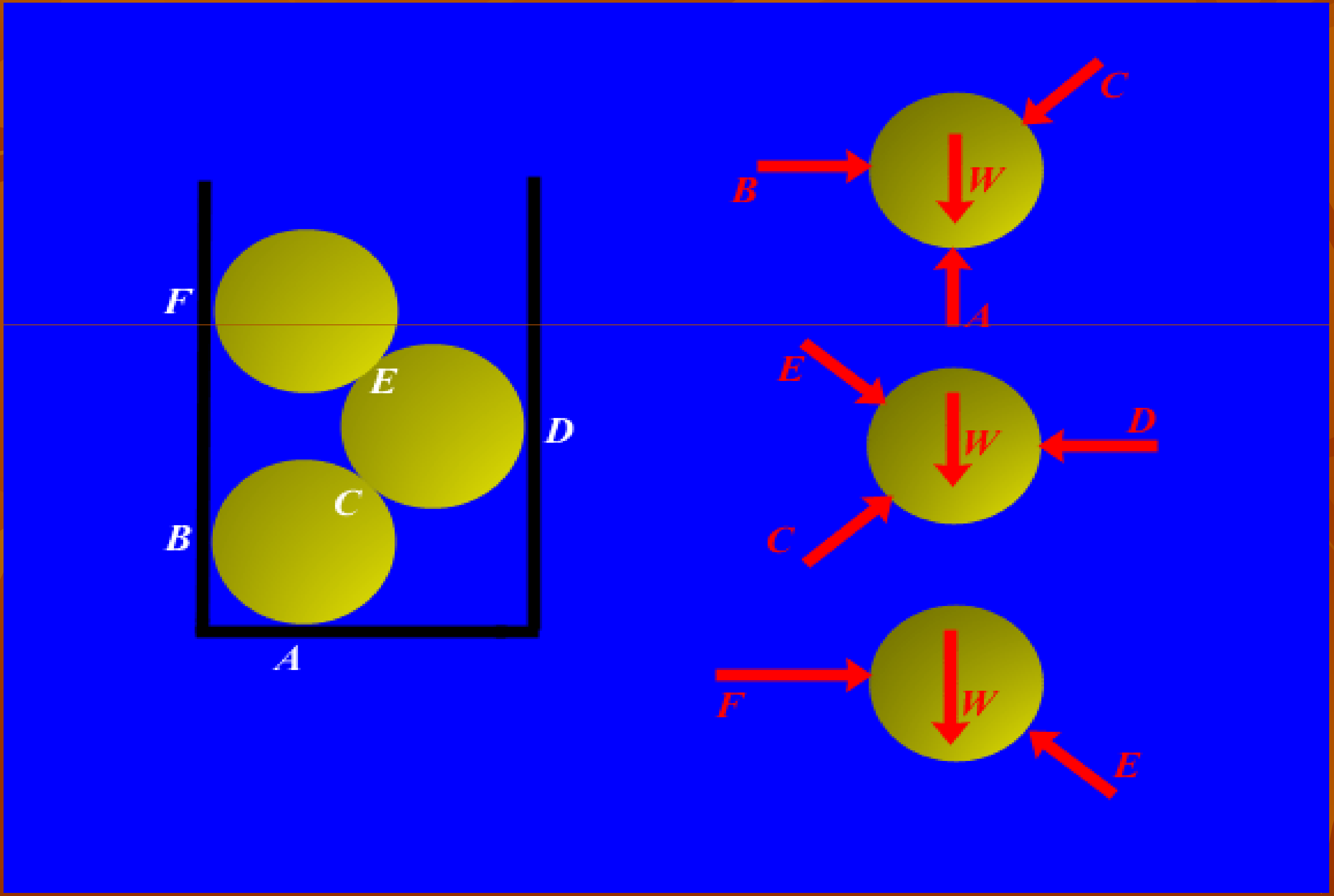
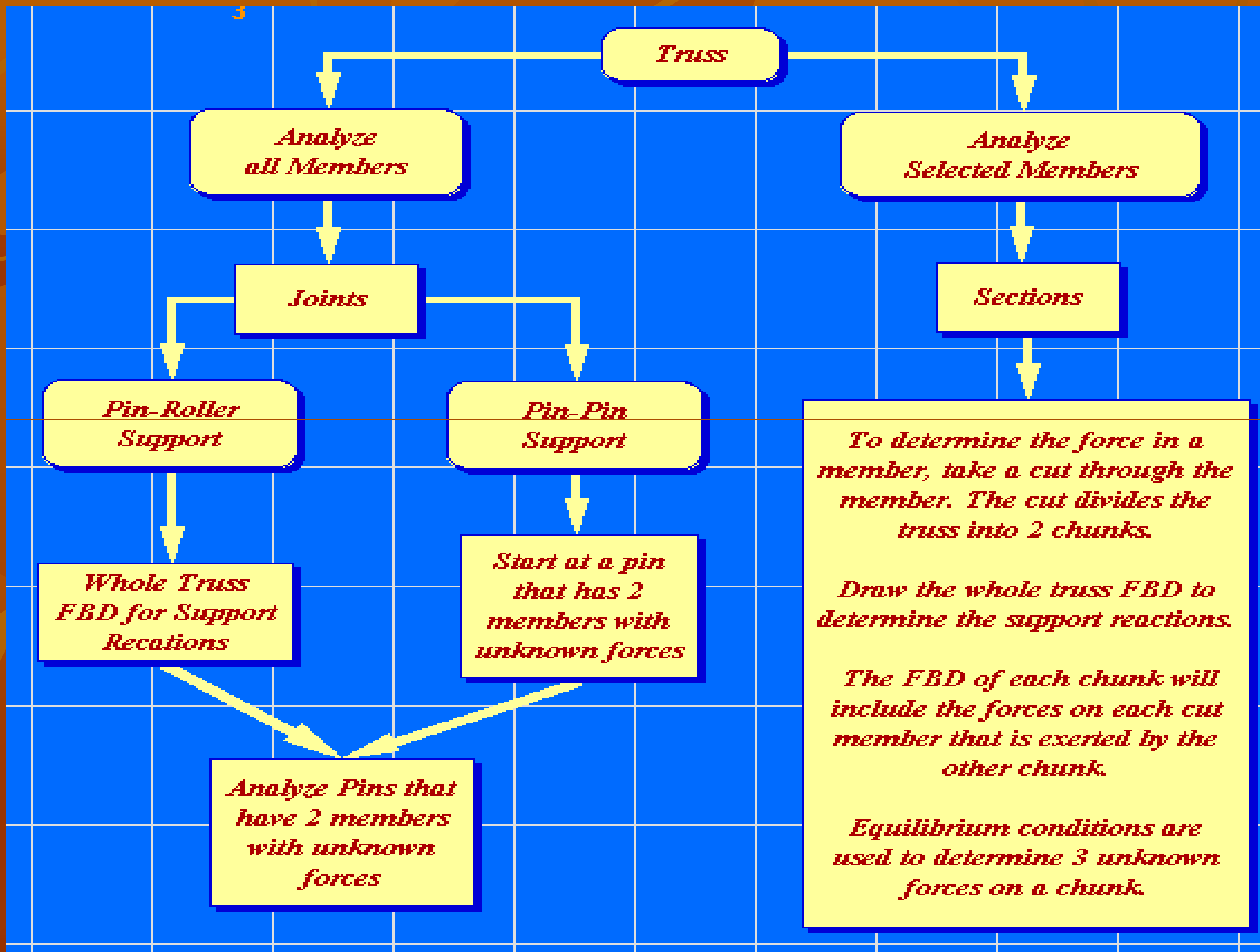
FBD Example-1



Truss Analysis

Contents



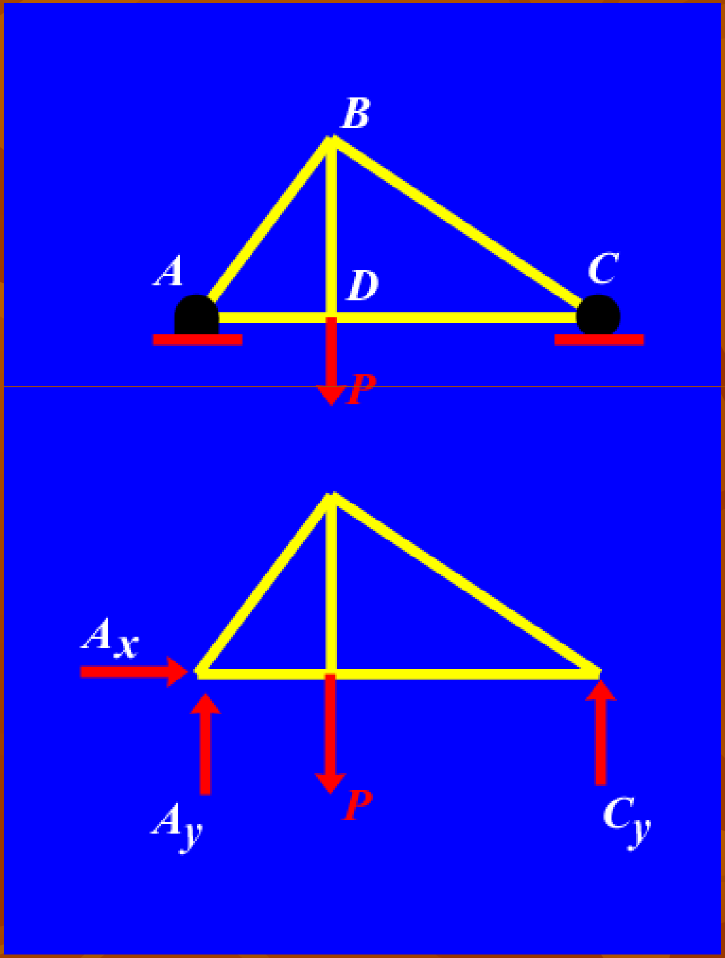
Contents

Truss

A structure consisting of 5 members, that are pinned at their ends, is supported by a pin at *A* and a roller at *C*.

All the members in this structure are 2-force and the structure is called a truss.

In the FBD of the entire structure we included only the forces arising from the supports



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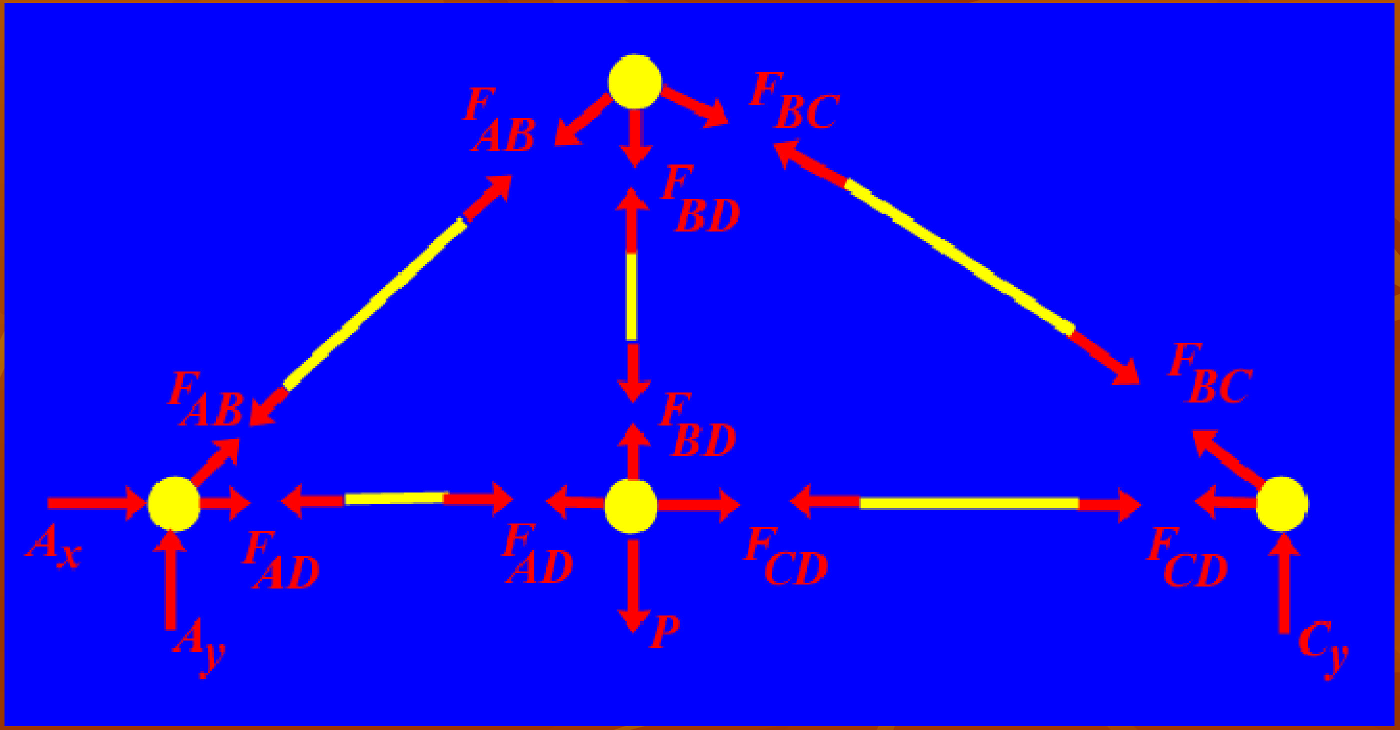
Method of Joints

The FBDs of each member and each pin in the truss is shown.

The forces in the members are obtained from the equilibrium of the pins that have concurrent loading.



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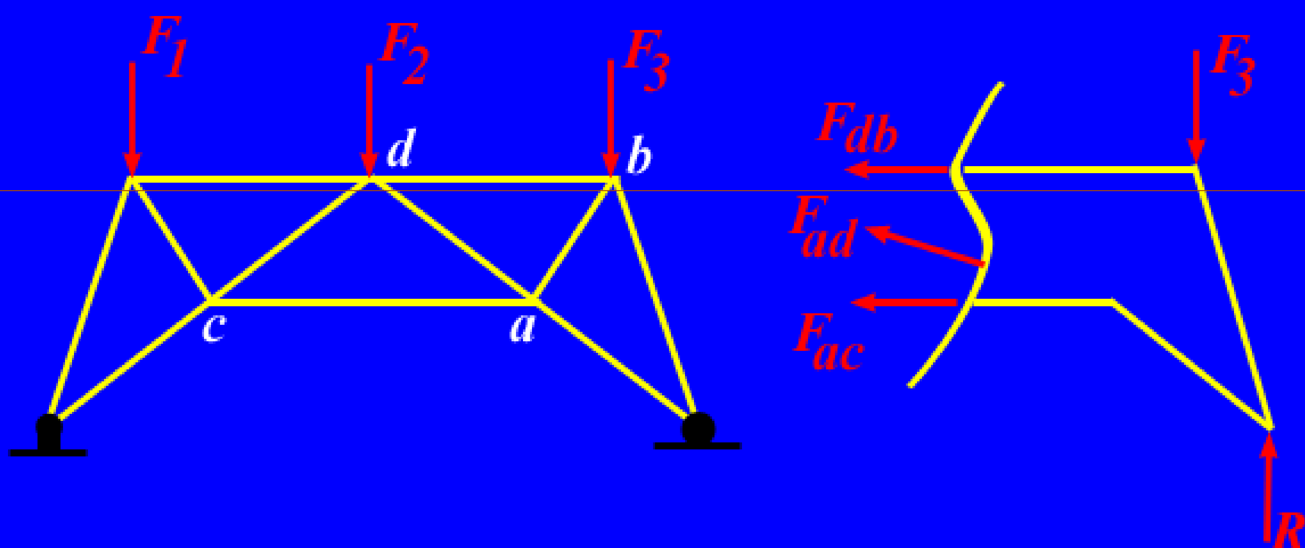
Method of Sections (1)

(a cut across 3 members)



Find forces in ac, ad, bd.

Write 3 equilibrium equations.

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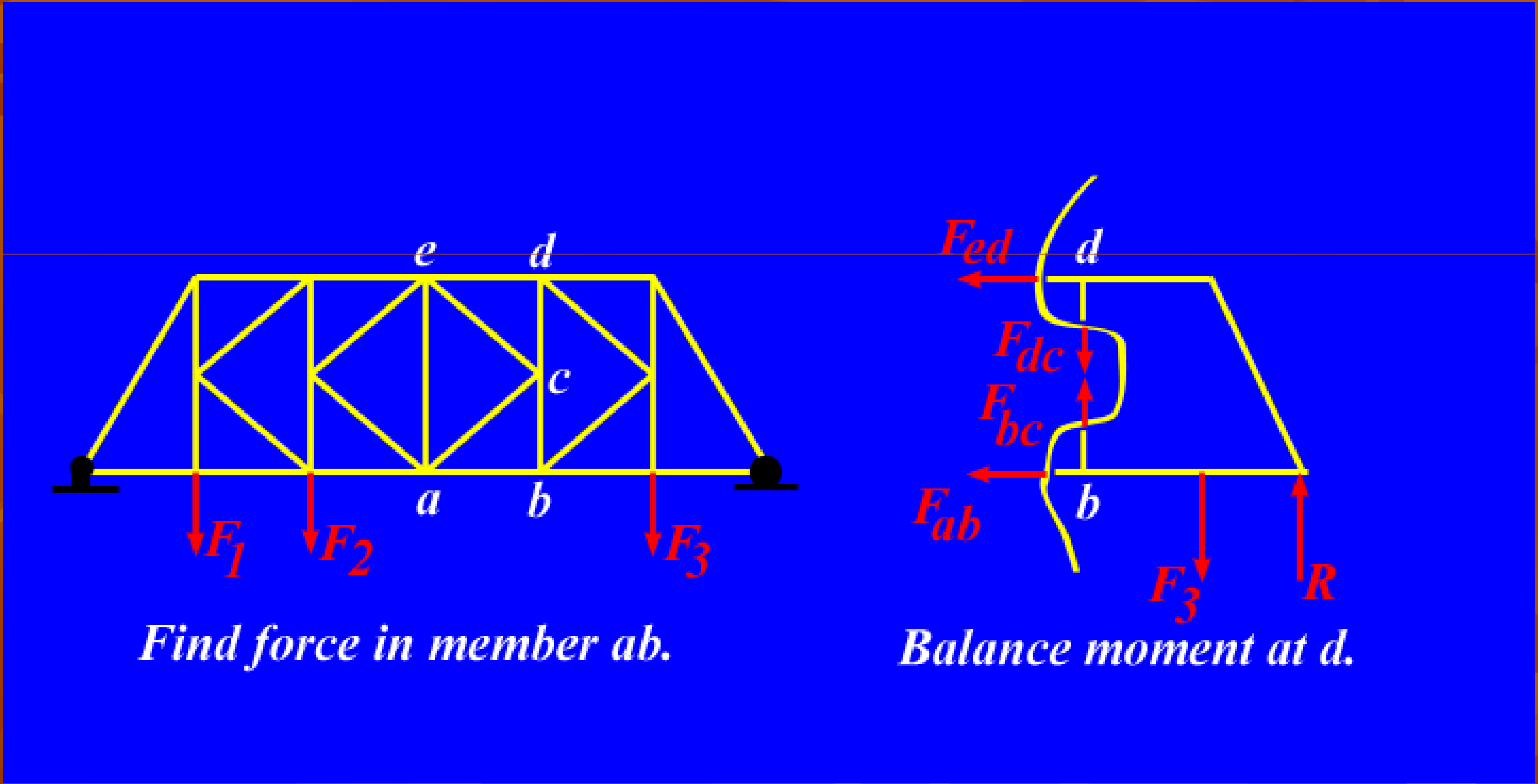
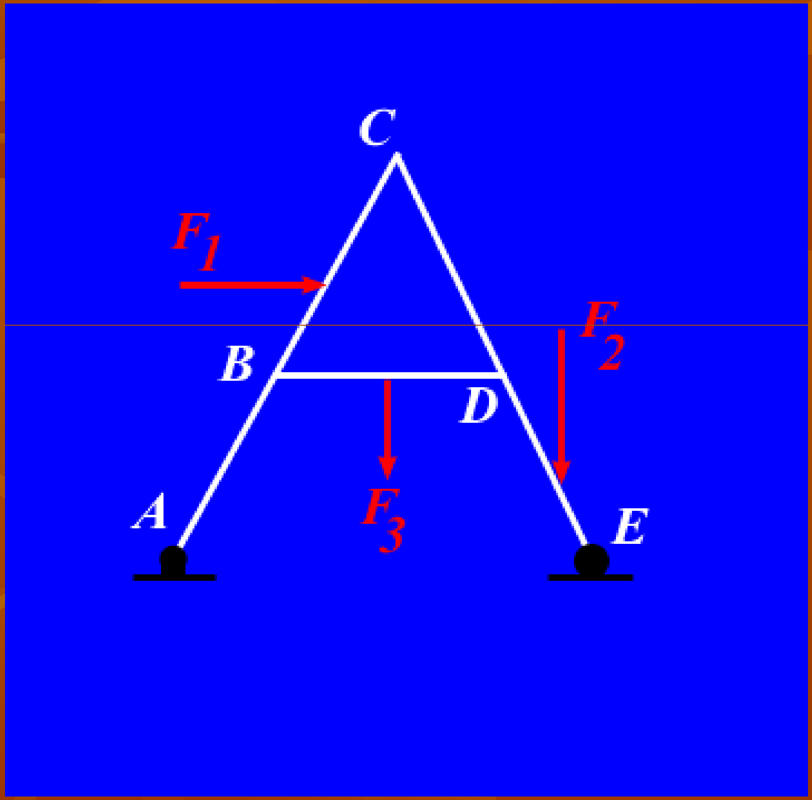
Method of Sections (2)

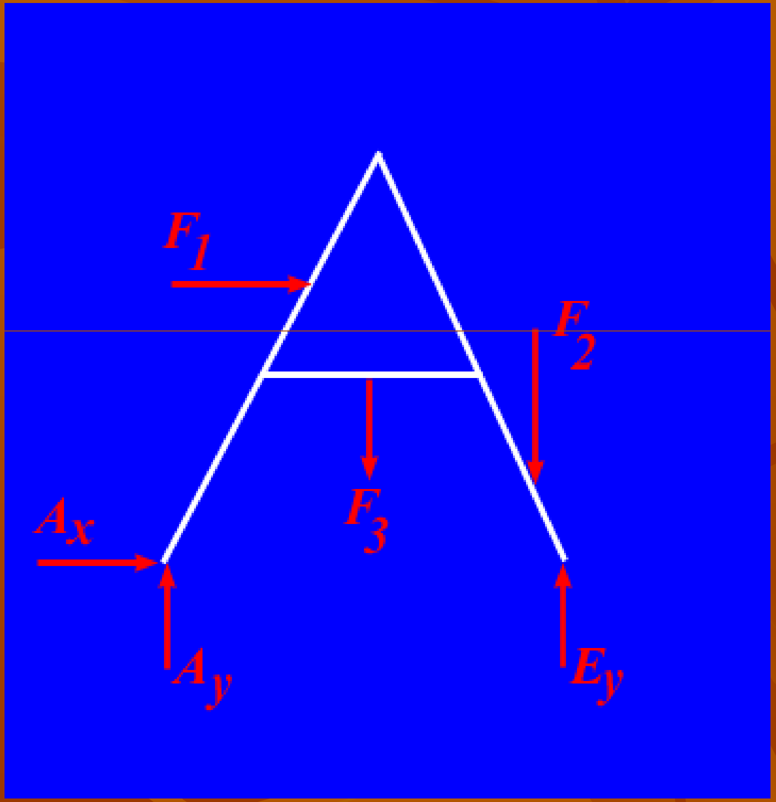
(a cut across 4 members)



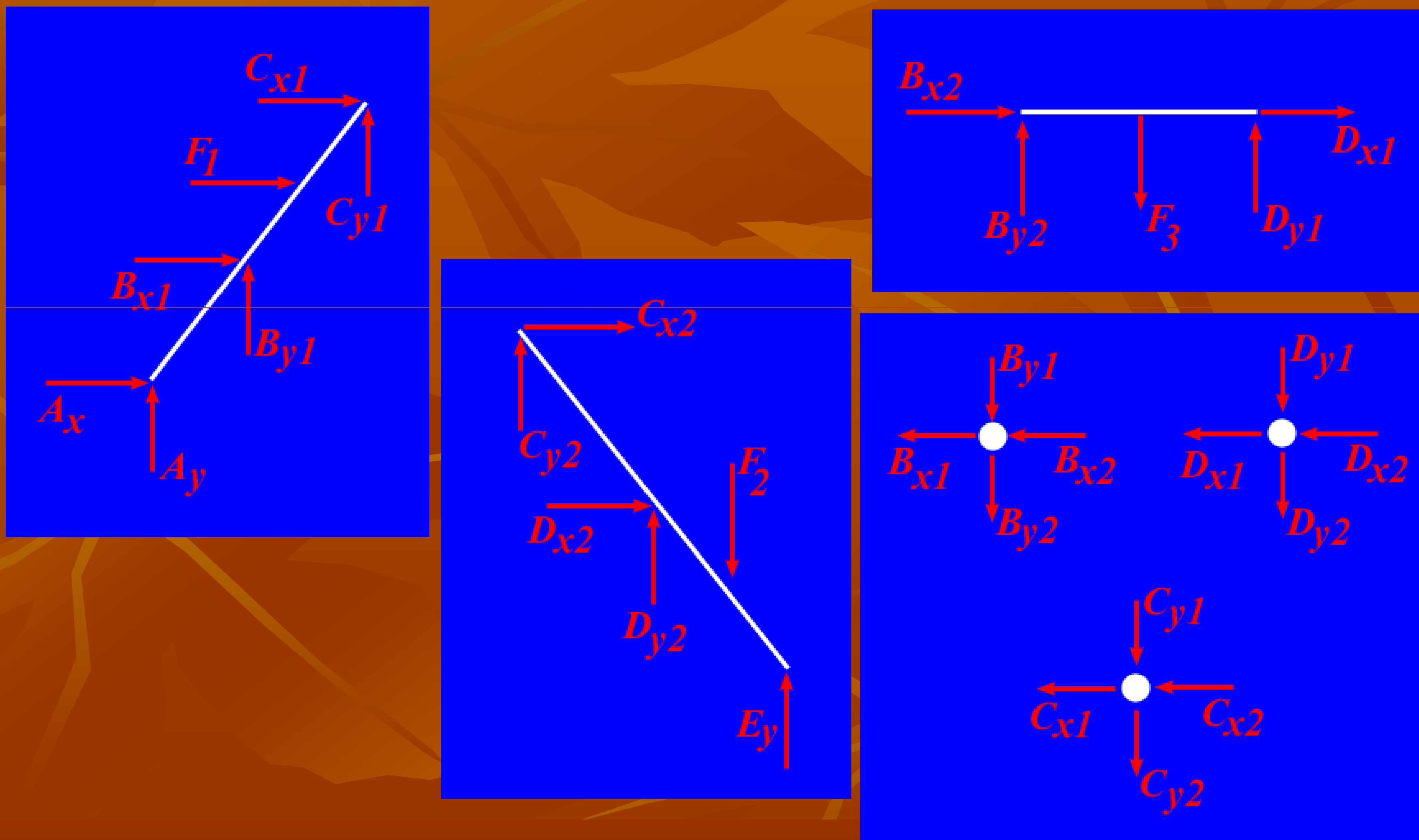
FBD Example-1
A-Frame

Contents





FBD Example-1 (contd.)



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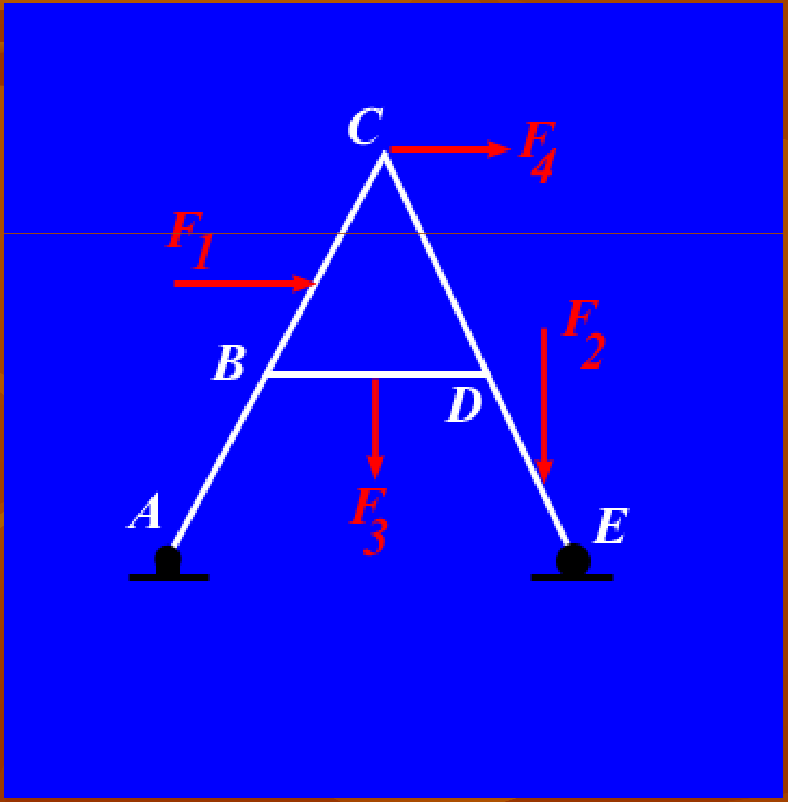
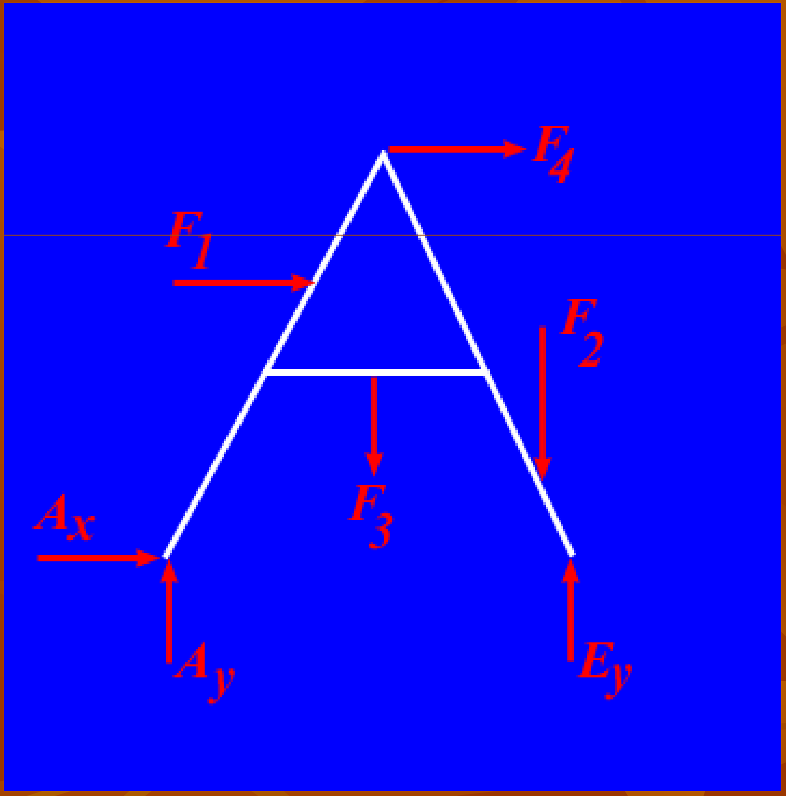
FBD Example-2
A-Frame with loaded pin C





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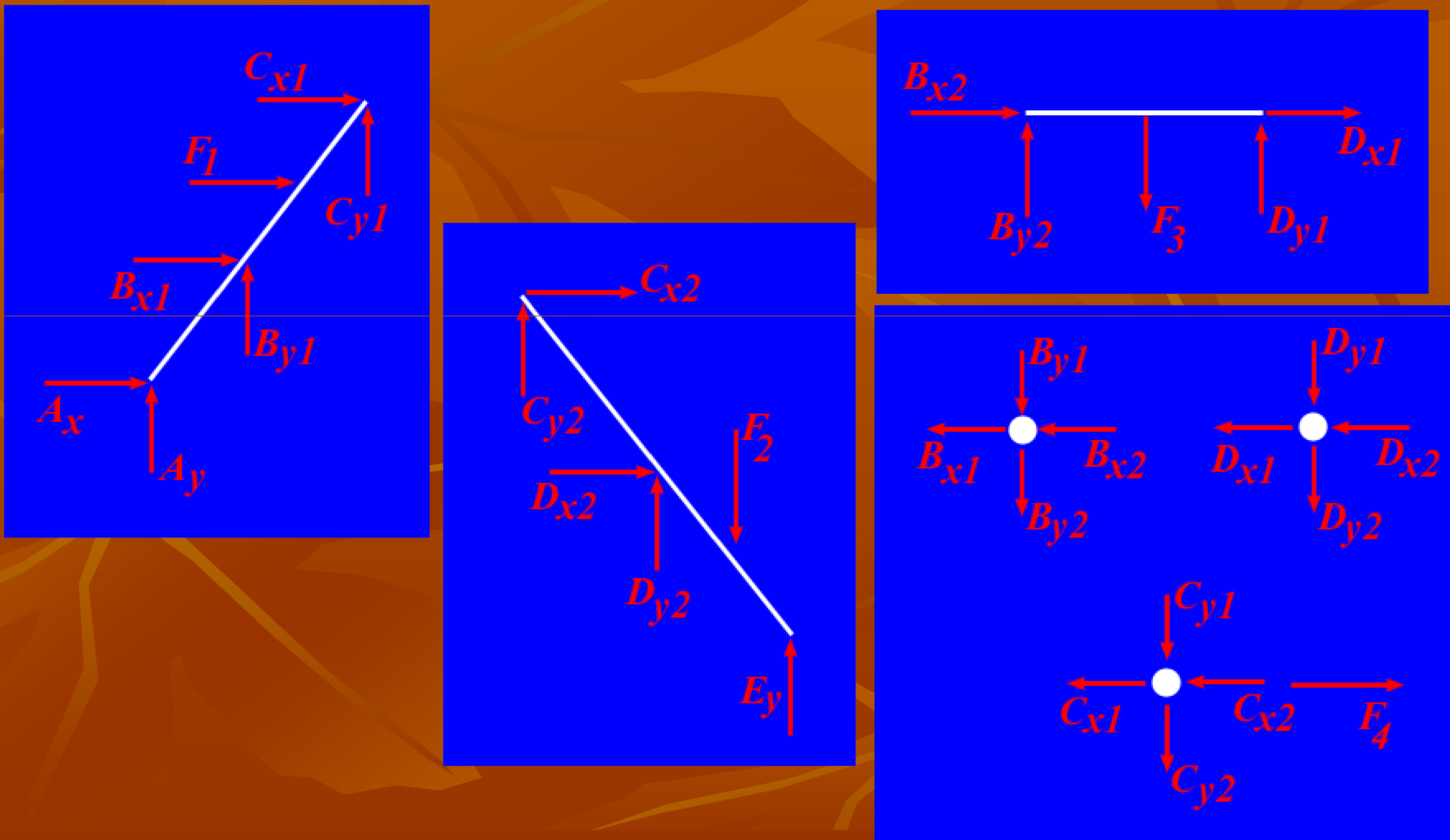
FBD Example-2 (contd.)

(note the load on Pin C)



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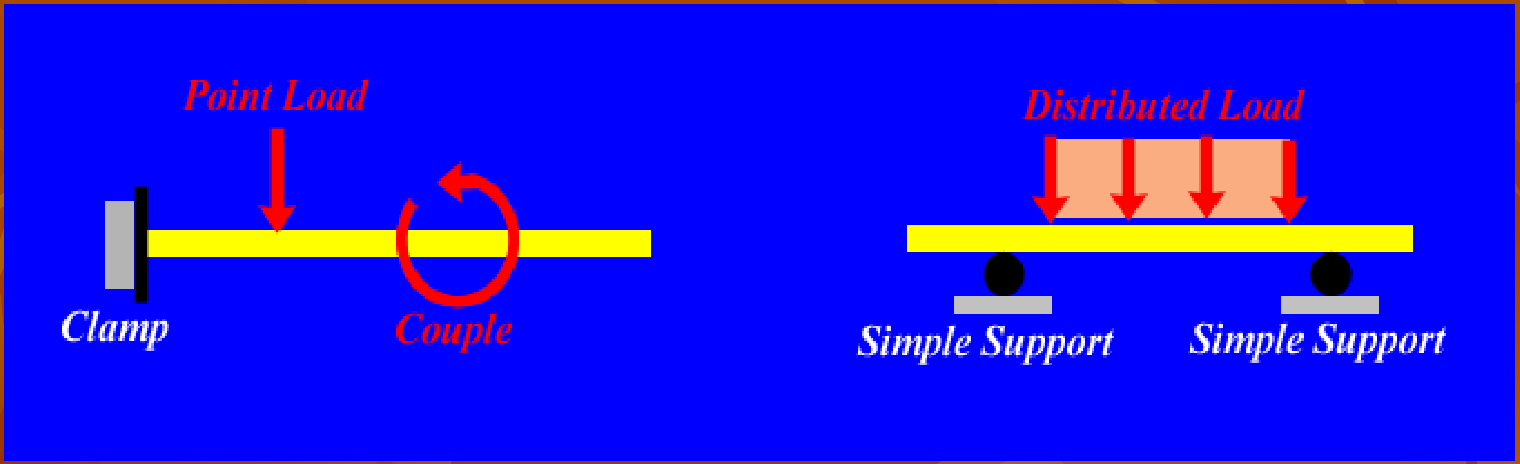
Beams

Beams are usually straight members that are loaded along a direction perpendicular to its length.

We will consider two types of beams: clamped and simply supported.

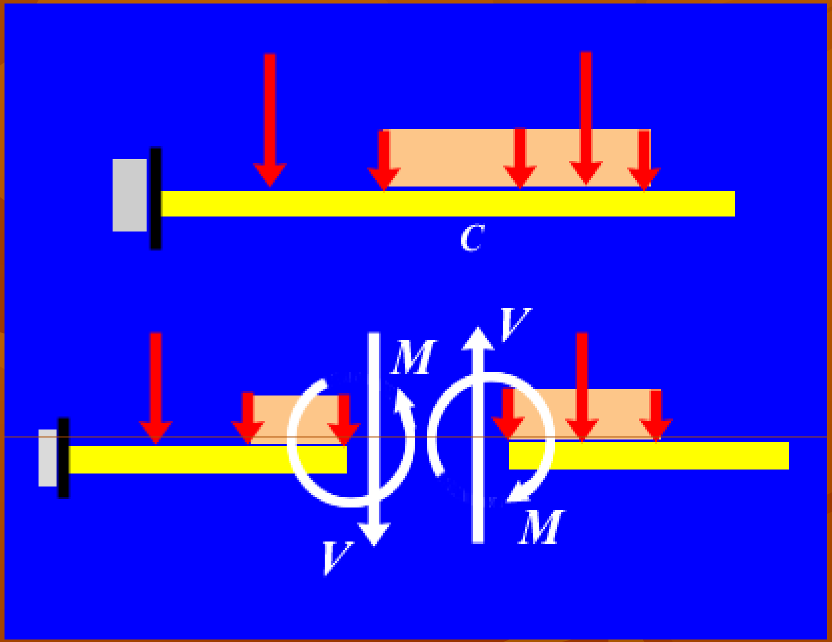
Loadings on a beam are: point forces, distributed forces, and couples.



Shear and Moment

Contents

To calculate the internal forces in the beam at section *C*, one needs to cut the beam at *C*. The cut exposes the shear force *V* and bending moment *M* at the cut.



Sign convention for *V*,*M*:

- *When the cut is on the right hand side, of a piece of the beam, V is downward and M is counter-clockwise.*
- *When the cut is on the left hand side, of a piece of the beam, V is upward and M is clockwise.*

Contents

Shear and Moment Diagrams

Shear Diagram

1. $V=0$ at the two ends of the beam.
2. Upward point loads give an upward jump to V.
3. Point moments have no effect on V.
4. Slope of V is equal to the value of the distributed load (w) at a point.
5. Constant w gives linear V; linear w gives quadratic V; etc.
6. Change in V between two stations $x1$ and $x2$ is

$$V_2 - V_1 = \int_{x1}^{x2} w \, dx$$

Moment Diagram

1. At the two ends of the beam $M=0$.
2. Clockwise point moments give an upward jump to M.
3. Point loads have no effect on M.
4. Slope of M is equal to the value of V at a point.
5. Constant V gives linear M; linear V gives quadratic M; etc.
6. M has max and min where V is zero.
7. Change in M between two stations $x1$ and $x2$ is

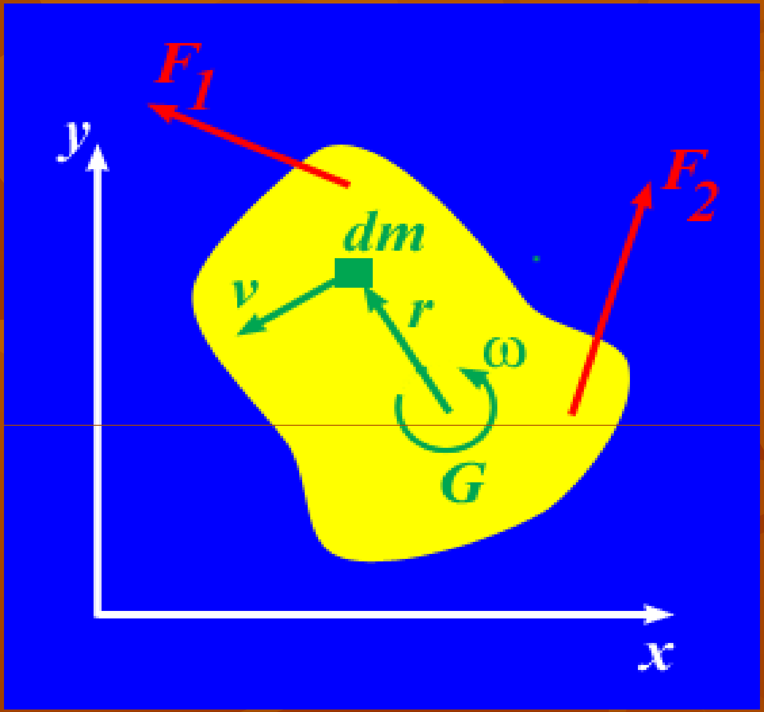
$$M_2 - M_1 = \int_{x1}^{x2} V \, dx$$

Plane Rotation, Moment of Inertia

Consider a plane rotational motion of a rigid slab, with center of mass at *G*, under the influence of several forces. The equation of motion is

$$\vec{M}_G = \frac{d\vec{L}_G}{dt}$$

M_G and L_G are moment and angular Momentum computed at *G*, and ω is the angular velocity.



$$\vec{L}_G = \int [\vec{r} \times \vec{v}]\, dm$$

$$= \int [\vec{r} \times (\omega \hat{k} \times \vec{r})]\, dm = \omega \left[\int r^2 dm\right] \hat{k} = I_G^{(m)} \omega \hat{k}$$

$I_G^{(m)}$ is the mass moment of inertia of the body.

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Area Moment of Inertia-2D

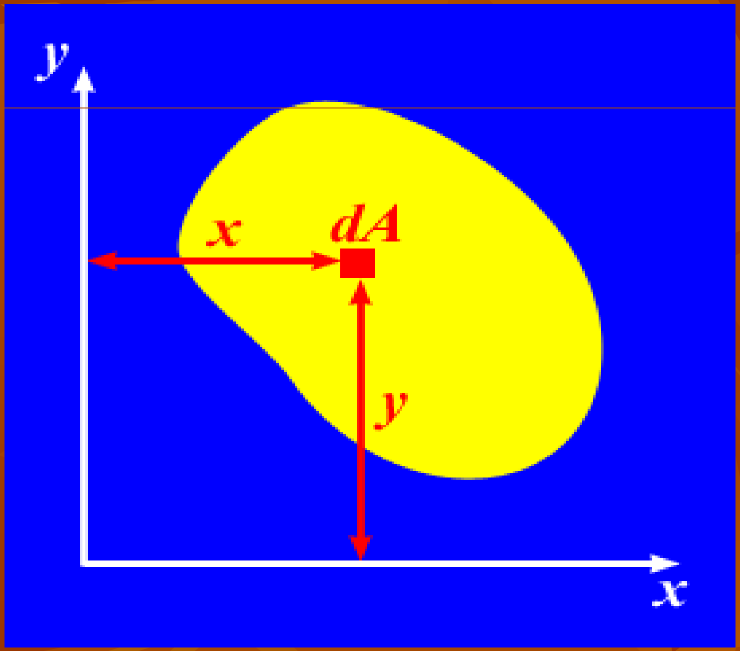
For a slab of uniform density

$$I_G^{(m)} = \int r^2 dm = \rho \int (x^2 + y^2) dA = \rho (I_{yy} + I_{xx})$$

The area moments of inertia are defined as

$$I_{xx} = \int y^2 dA = \int y^2 dxdy$$

$$I_{yy} = \int x^2 dA = \int x^2 dxdy$$



The product of inertia is defined as

$$I_{xy} = \int xydA = \int xydxdy$$